Distance Properties of Short LDPC Codes and Their Impact on the BP, ML and Near-ML Decoding Performance

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- Norwegian-Estonian Research Cooperation Programme (grant EMP133)
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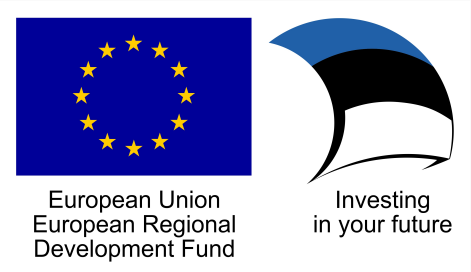

Outline

1. Code parameters
2. Stopping redundancy hierarchy
3. Considered codes
4. Simulations: FER performance
5. Spectra and bounds
6. But what about BAWGN channel?

Code parameters

Code parameters influence on decoding success (BEC)

Decoding problem on BEC

Solve linear system: $H\,\mathbf{x}^\intercal = \mathbf{0}^\intercal$ for $\mathbf{x} = (x_1, x_2, x_3, x_4, ?, x_6, x_7, ?, x_9, ?)$

[1] BP with extended parity-check matrix

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Table 1: Linear $[n, k, d_{\min}]$ code and its parameters

parameter	Decoding algorithm		
	BP (belief propagation)	Near-ML[1]	Maximum-likelihood (ML)
$d_{\min}$, distance spectrum	✓		✓
d_{stop}, stopping spectrum	✓		
d_{dual}		✓	
girth g	✓	✓	
SR hierarchy		✓	

[1] BP with extended parity-check matrix

Stopping redundancy hierarchy

Stopping sets and d_{stop}

$$H = \begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & ? & x_6 & x_7 & ? & x_9 & ? \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & 1 & \mathbf{1} \\ 1 & 1 & 0 & 1 & \mathbf{1} & 0 & 0 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 0 & 1 & 0 & \mathbf{0} & 1 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 1 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 1 & \mathbf{1} \\ 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{0} \\ 1 & 0 & 0 & 1 & \mathbf{0} & 0 & 1 & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

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Stopping distance, d_{stop}

Size of the smallest stopping set.

Stopping redundancy hierarchy (by example)

Aim

By adding redundant rows, remove small stopping sets (up to size ℓ)

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$$
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\\
c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & ? & x_6 & x_7 & ? & x_9 & ? \end{array} \\
\left(\begin{array}{cccccccccc}
1 & 1 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & 1 & \mathbf{1} \\
1 & 1 & 0 & 1 & \mathbf{1} & 0 & 0 & \mathbf{0} & 1 & \mathbf{1} \\
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\end{array}\right)
\end{array}
$$

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$$H' = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ \hline c_1+c_2+c_3 \\ c_2+c_3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array} \\ \left(\begin{array}{cccccccccc} 1 & 1 & 0 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & 1 & \mathbf{1} \\ 1 & 1 & 0 & 1 & \mathbf{1} & 0 & 0 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 0 & 1 & 0 & \mathbf{0} & 1 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 1 & 1 & \mathbf{1} & 0 & 1 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 1 & \mathbf{1} \\ 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{0} \\ 1 & 0 & 0 & 1 & \mathbf{0} & 0 & 1 & \mathbf{1} & 0 & \mathbf{1} \\ \hline 1 & 0 & 1 & 0 & \mathbf{0} & 1 & 1 & \mathbf{0} & 0 & \mathbf{1} \\ 0 & 1 & 1 & 1 & \mathbf{1} & 1 & 0 & \mathbf{1} & 1 & \mathbf{0} \end{array}\right) \end{array}$$

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ℓ-th stopping redundancy

Minimum number of rows ρ_ℓ, s.t. there are no stopping sets of size up to ℓ (except codewords).

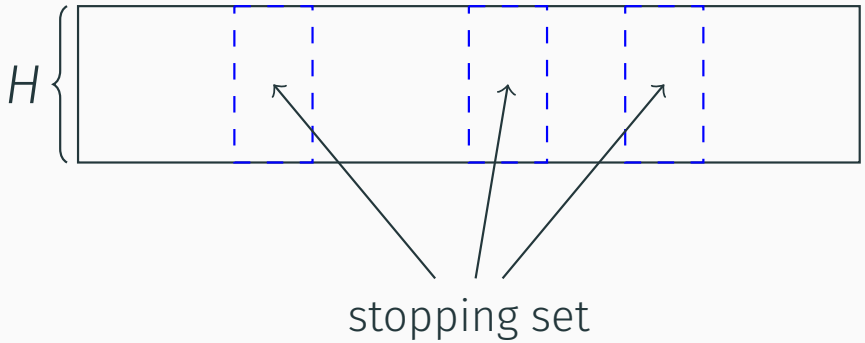
Achieving ML performance

Corrollary from definition

It's possible to build $\rho_{n-k} \times n$ extended parity-check matrix H', s.t. BP decoder **achieves ML performance**.

Stopping redundancy – upper bound (i)

Intuition/main observation



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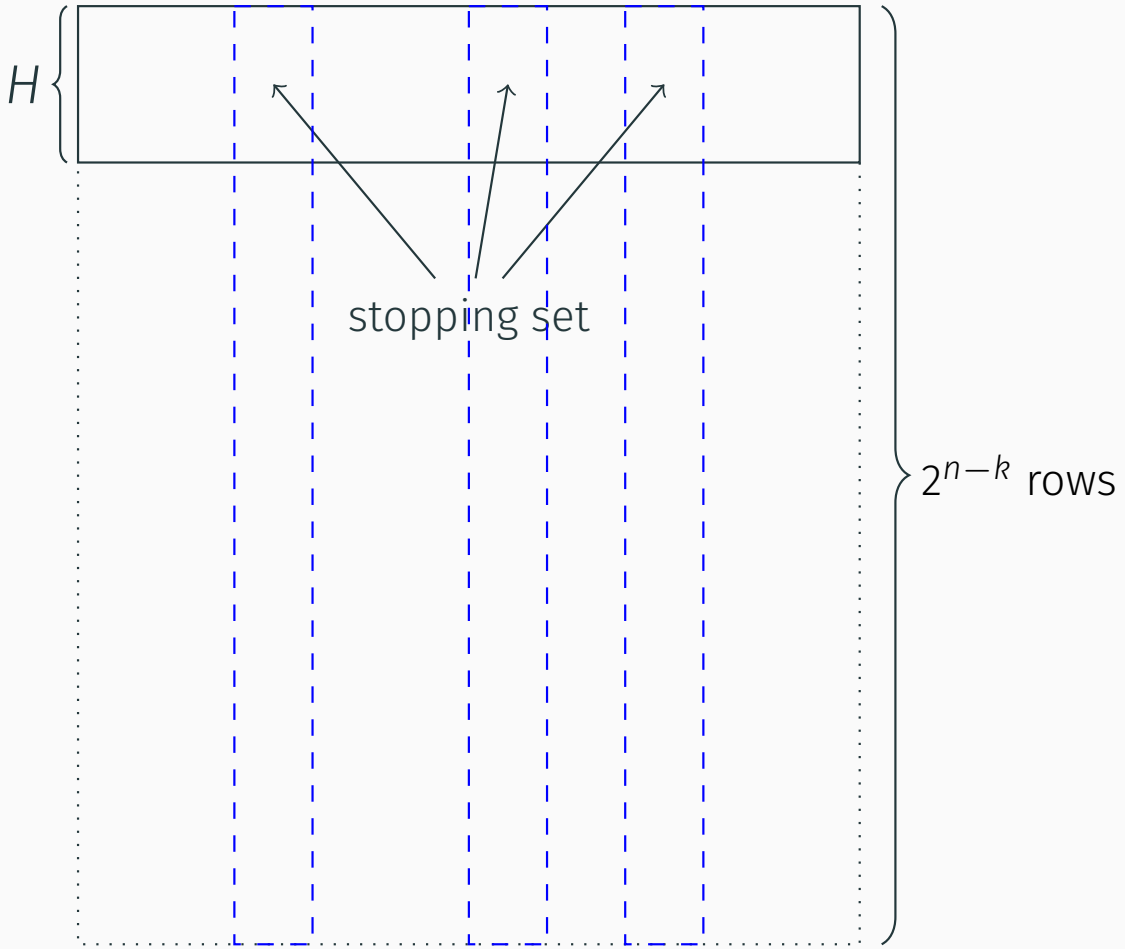

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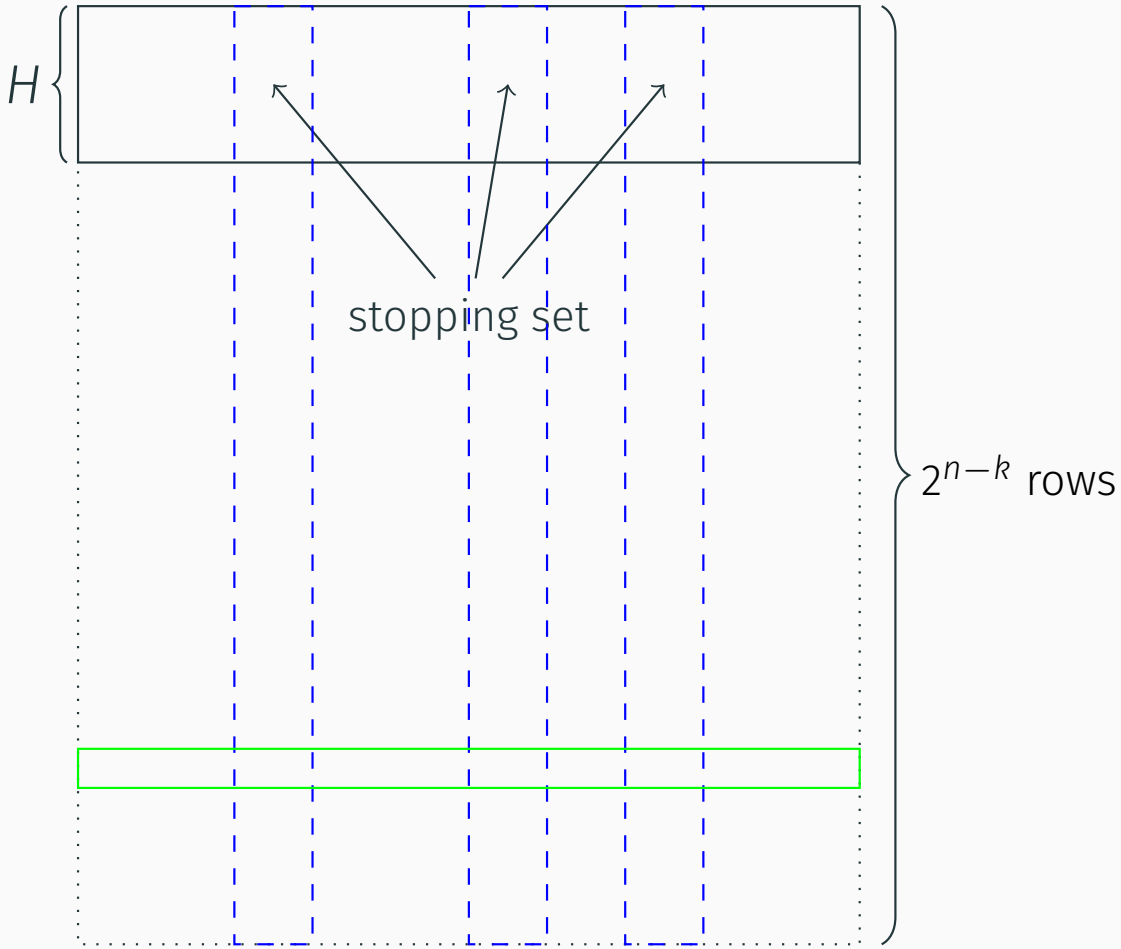

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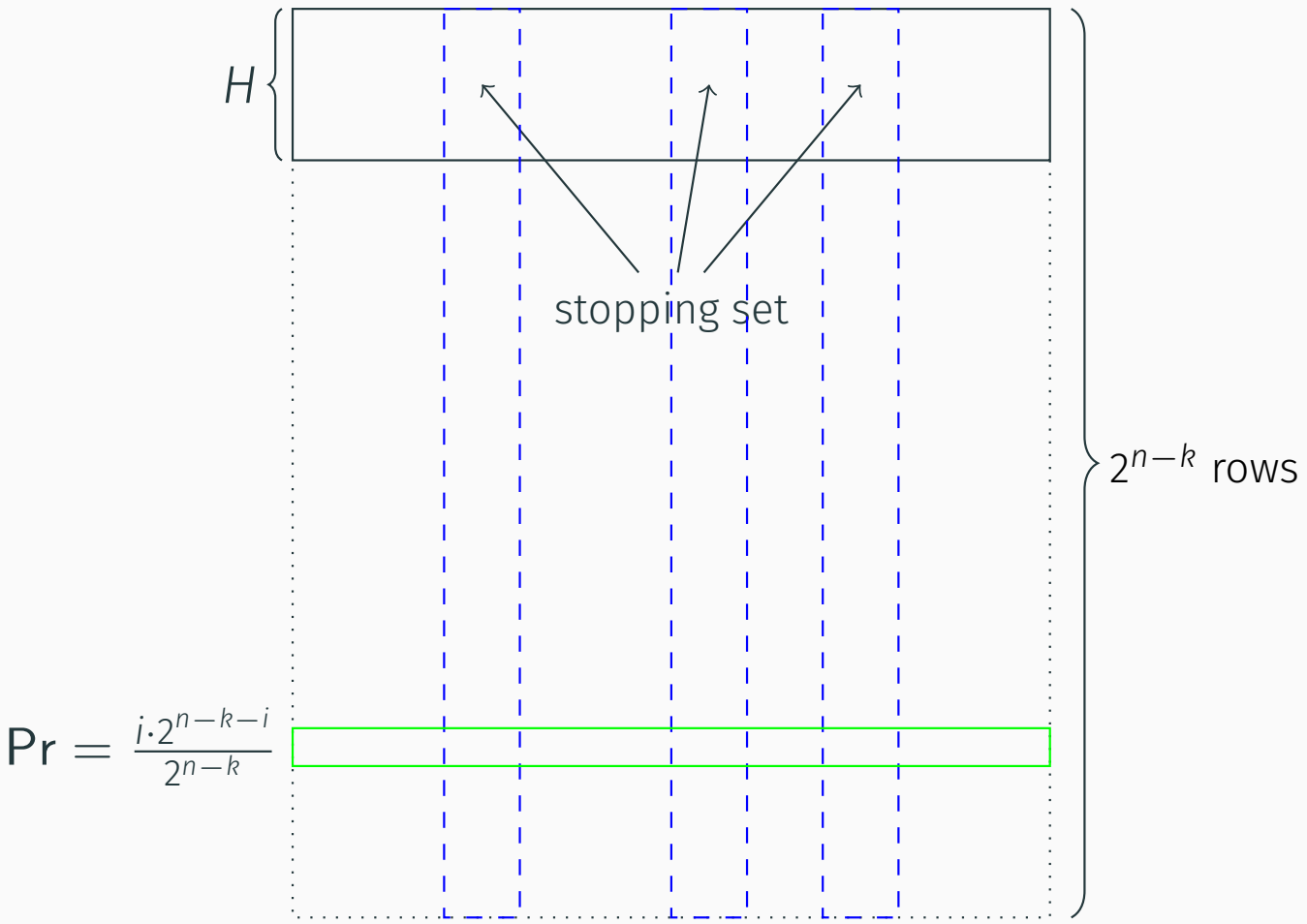

Stopping redundancy – upper bound (ii)

Upper bound construction

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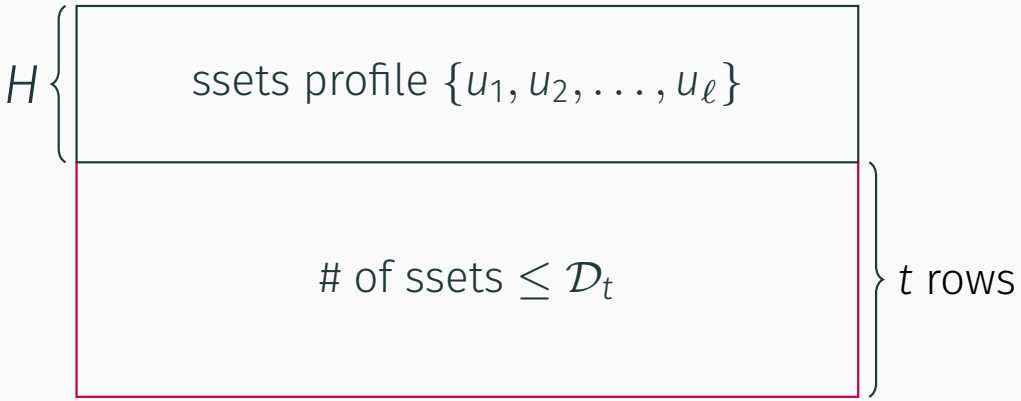

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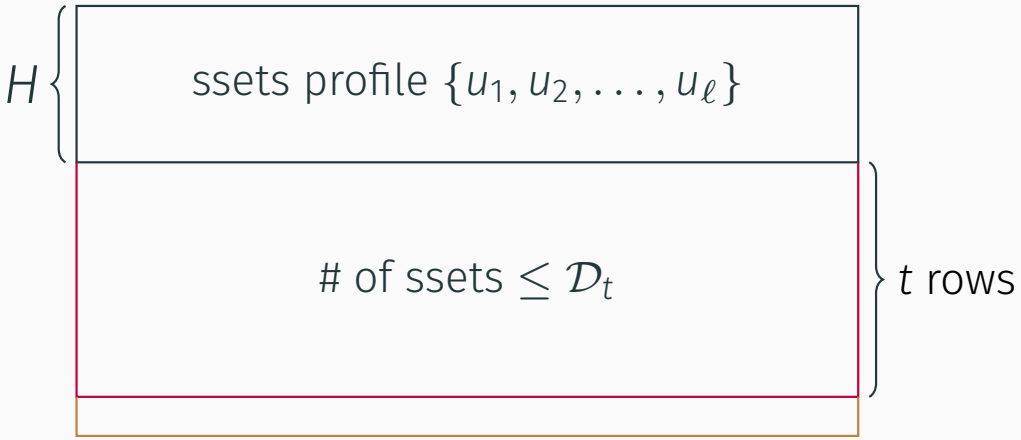

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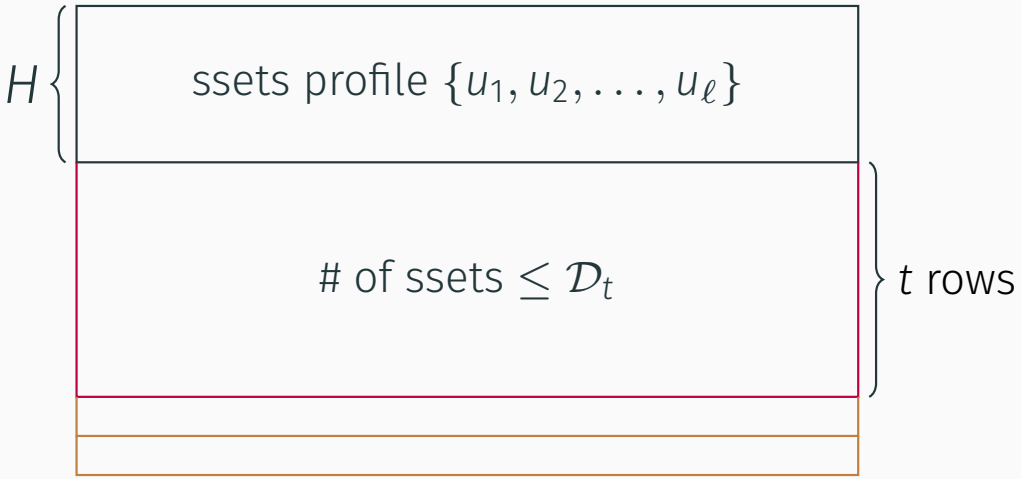

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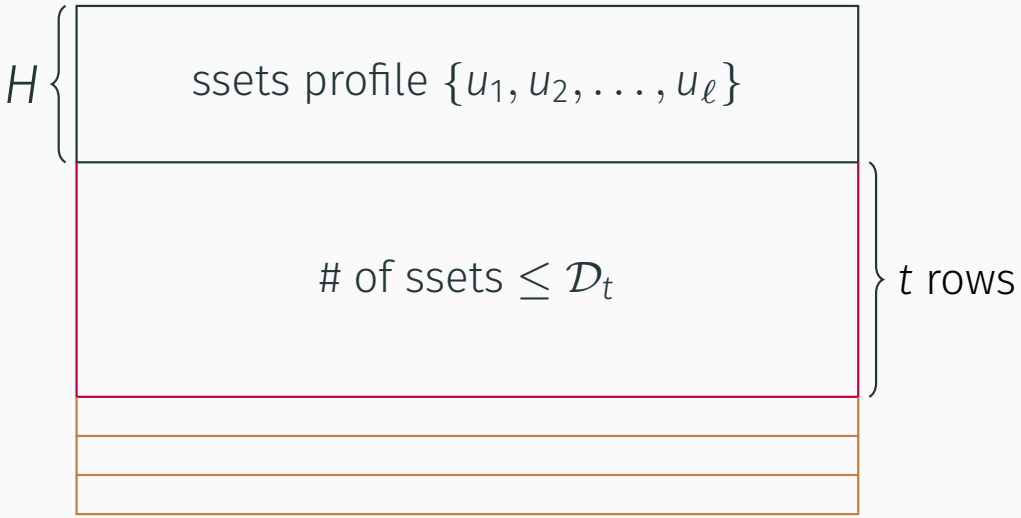

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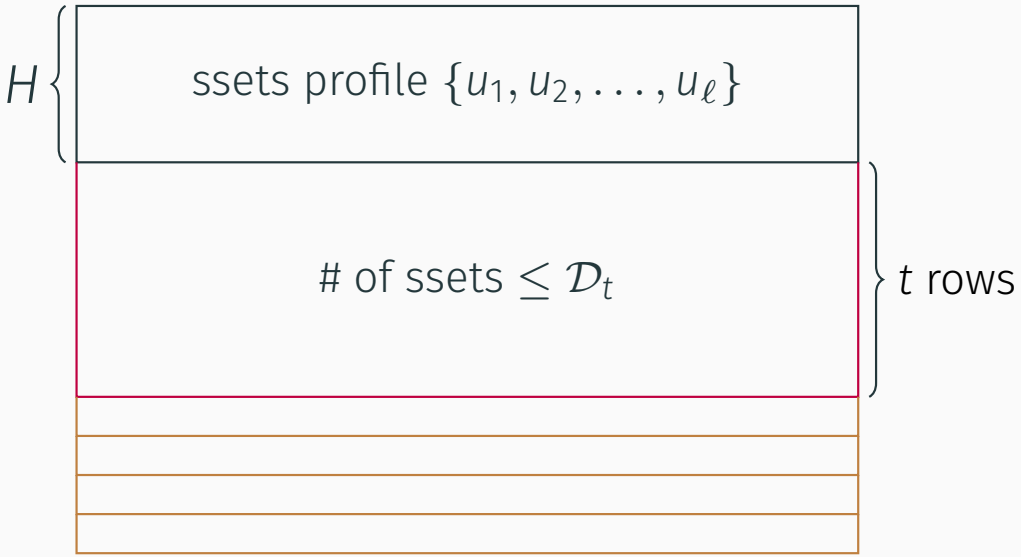

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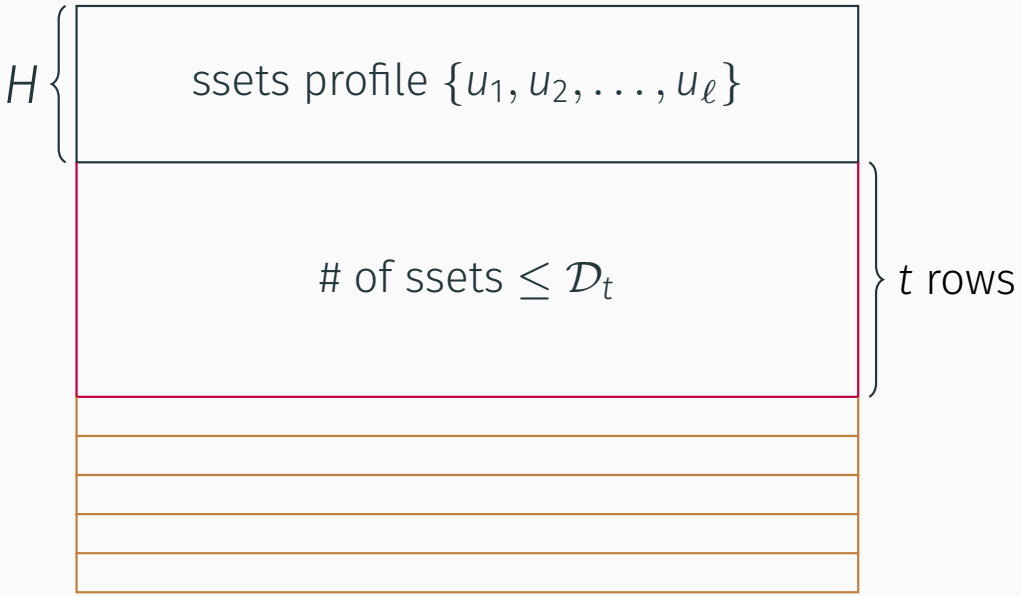

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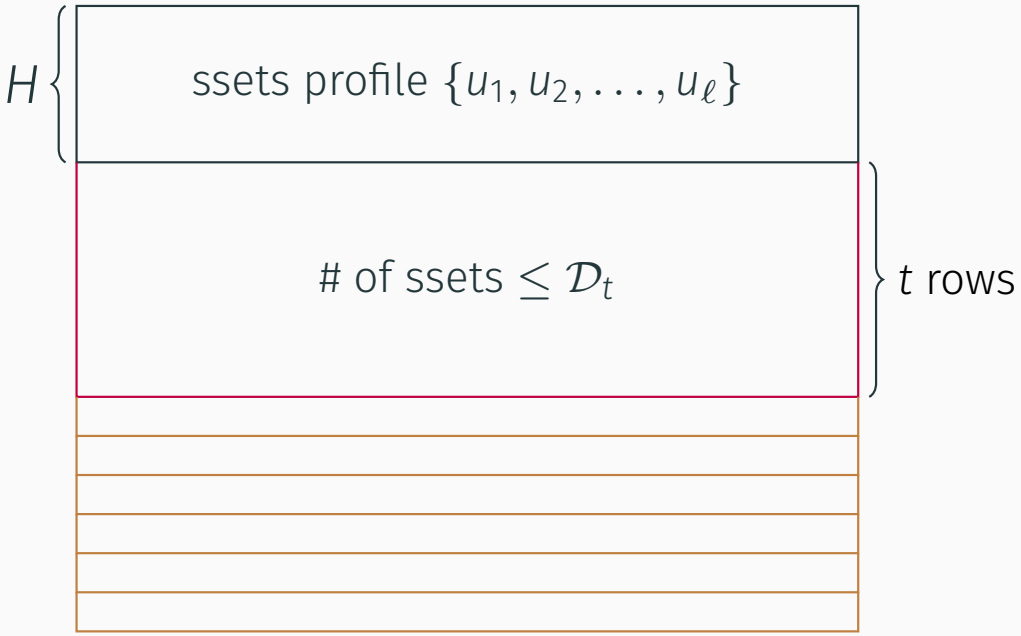

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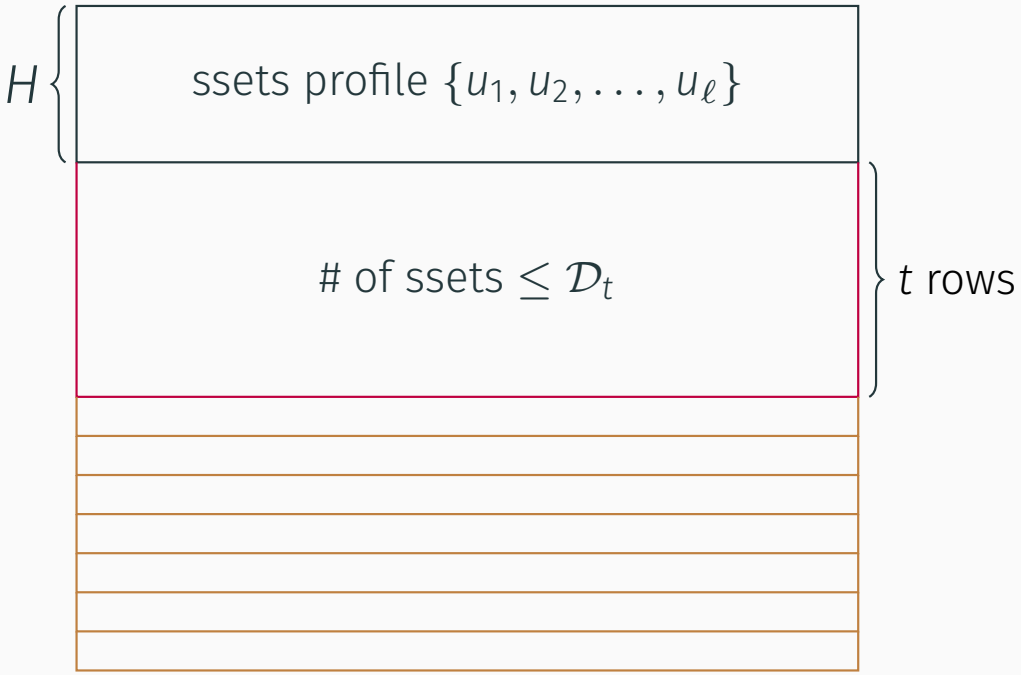

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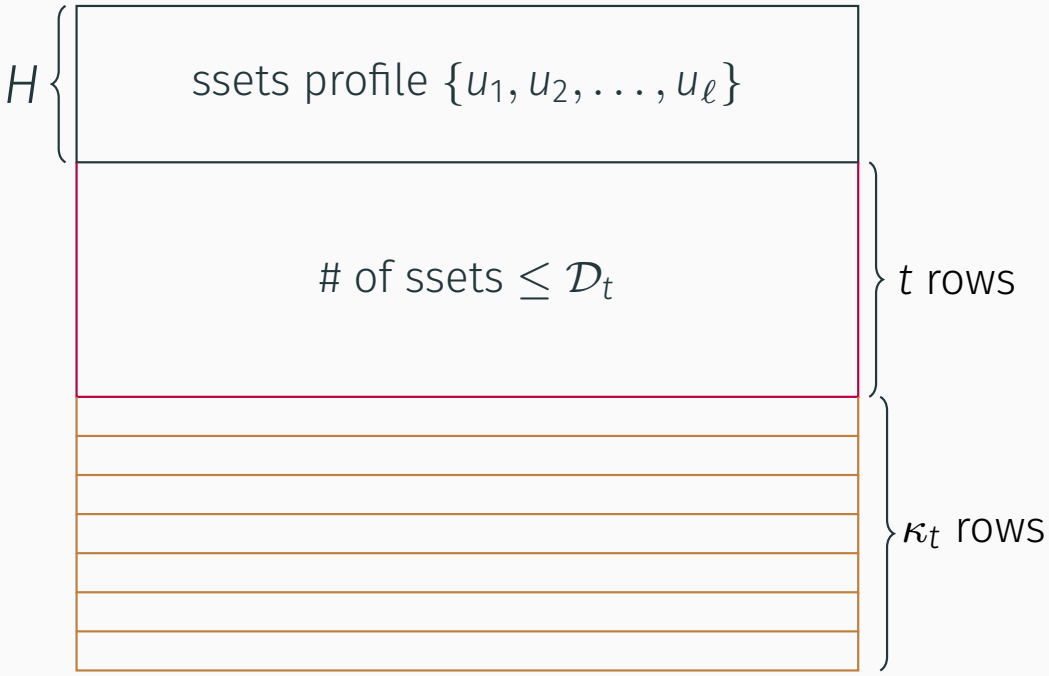


Stopping redundancy – upper bound (ii)

Upper bound[2,3] on stopping redundancy

$$\rho_\ell \leq n - k + \min_{t \in \mathbb{N}} \{t + \kappa_t\}$$

where

$$\mathcal{D}_t = \sum_{i=1}^{\ell} u_i \prod_{j=n-k+1}^{n-k+t} \left(1 - \frac{i \cdot 2^{n-k-i}}{2^{n-k} - j}\right)$$

$$P_{t,j}(x) = \left\lfloor x \left(1 - \frac{\ell \cdot 2^{n-k-\ell}}{2^{n-k} - (n-k+t+j)}\right) \right\rfloor$$

$$\kappa_t = \min \left\{ j \in \mathbb{N} : P_{t,j}(P_{t,j-1}(\ldots P_{t,1}(\lfloor \mathcal{D}_t \rfloor)\ldots)) = 0 \right\}$$

---

[2] Han, Siegel, Vardy. (2008). Improved probabilistic bounds on stopping redundancy.

[3] Yakimenka, Skachek. (2015). Refined upper bounds on stopping redundancy of binary linear codes.

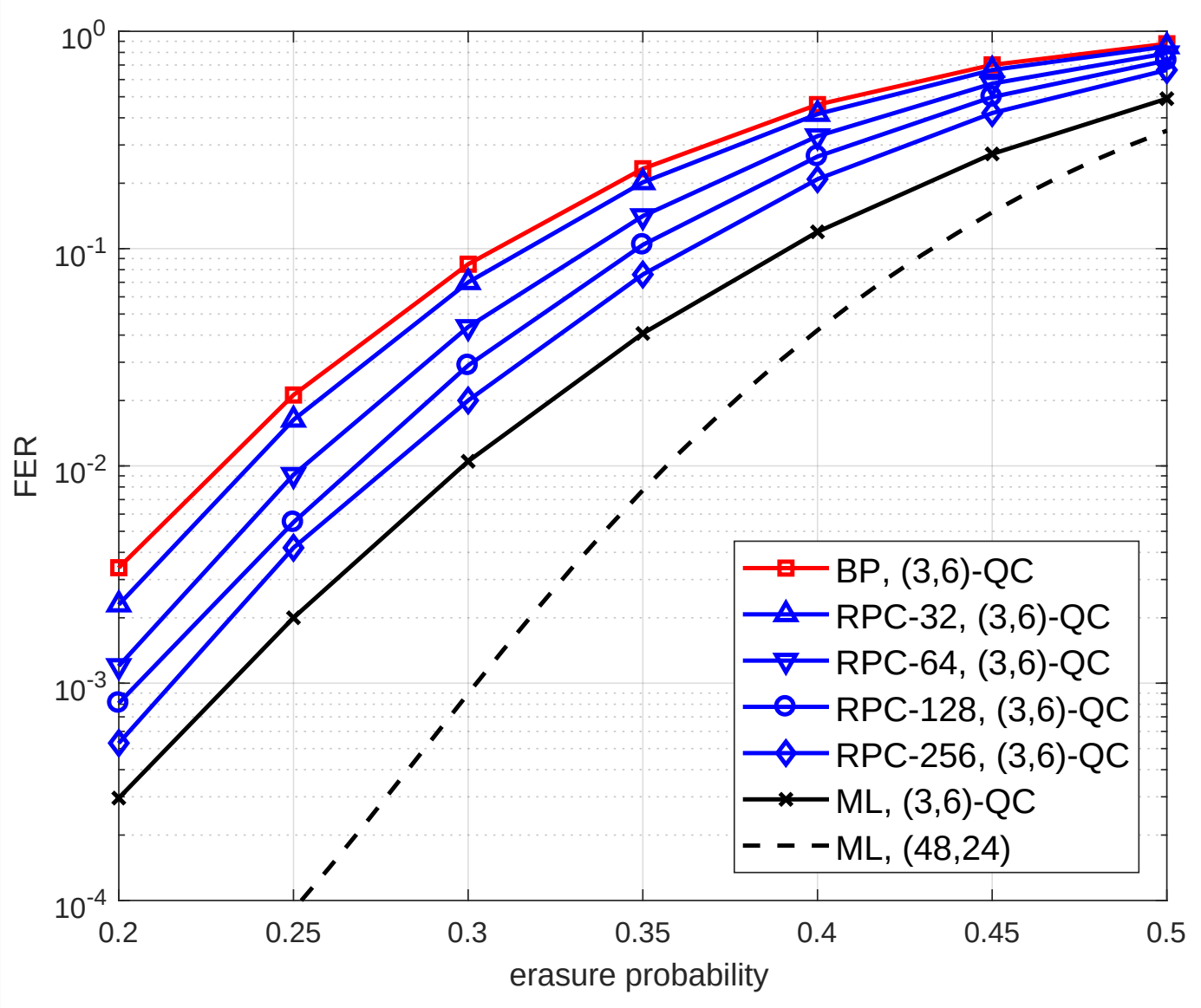
Considered codes

Parameters of studied $[48, 24]$-codes

$d_{\min}$	$A_{d_{\min},n}$	d_{stop}	d_{dual}	g	(J, K)	$\rho_{d_{\min}}, \rho_{d_{\min}+1}$	ρ_r	Type
12	**17296**	**4**	**12**	**4**	$(6, 12)$	**6240,12151**	$\mathbf{1.38 \times 10^7}$	**'L'**
8	13	4	6	4	$(6, 12)$	261,581	1.37×10^7	'RU'
7	1	5	5	4	$(4, 8)$	83,175	1.25×10^7	'RU'
7	**8**	**7**	**5**	**6**	$(3, 6)$	**58,130**	$\mathbf{0.99 \times 10^7}$	**'QC'**
8	**7**	**4**	**7**	**4**	$(3, 6)$	**355,751**	$\mathbf{1.38 \times 10^7}$	**'NB'**

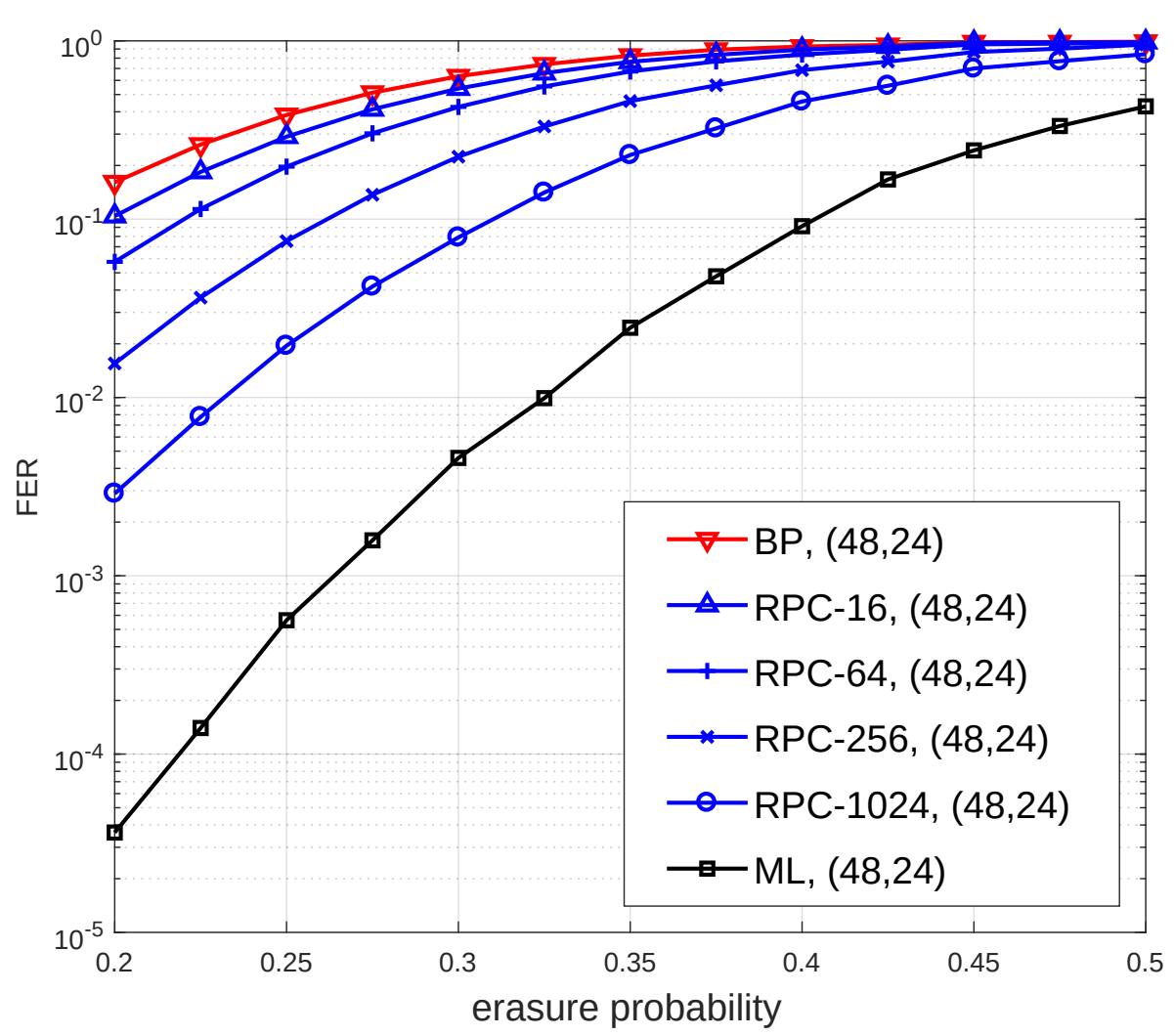
Simulations: FER performance

QC $(3, 6)$-regular code



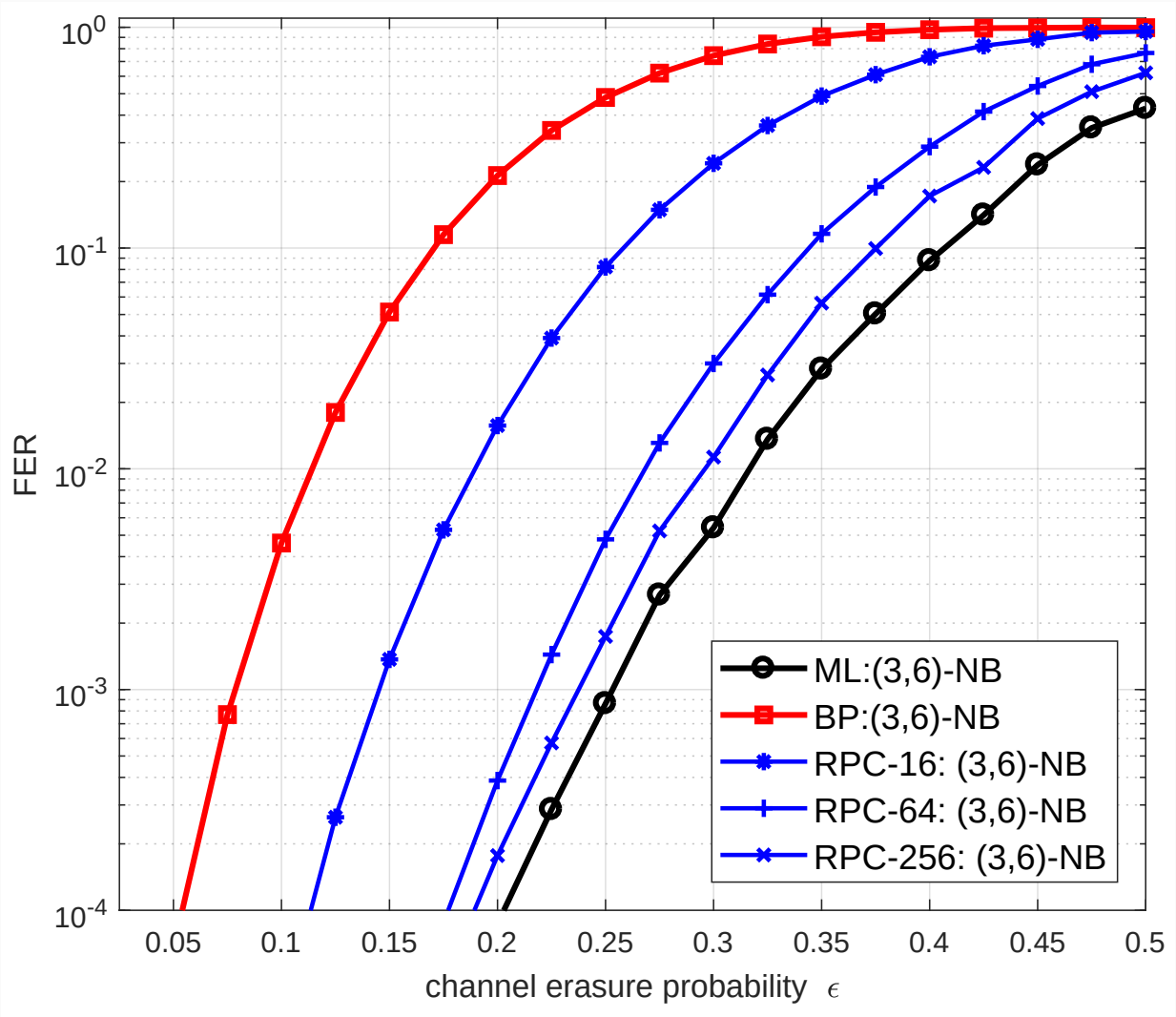
Good convergence to ML, but ML performance is poor

Linear $[48, 24, 12]$ code



RPC is efficient enough, but convergence to ML is slow

Binary image of non-binary $(3, 6)$-code



Both convergence and ML performance are good

Spectra and bounds

Spectra

We consider the following spectra for ensembles:

- Distance spectra
- Stopping sets spectra
- Stopping sets spectra for binary images of non-binary codes (**gave us bounds for BP decoding**)

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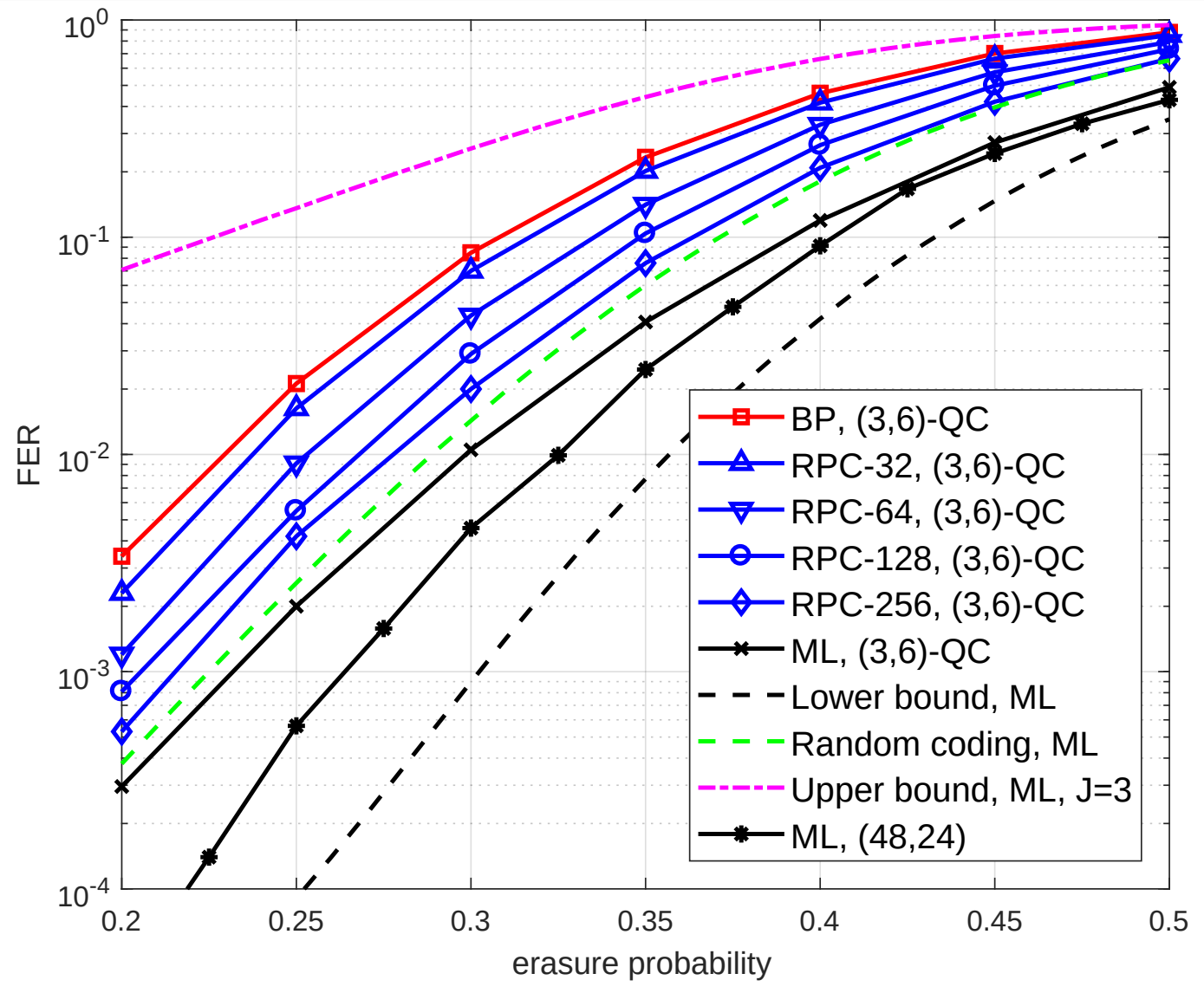

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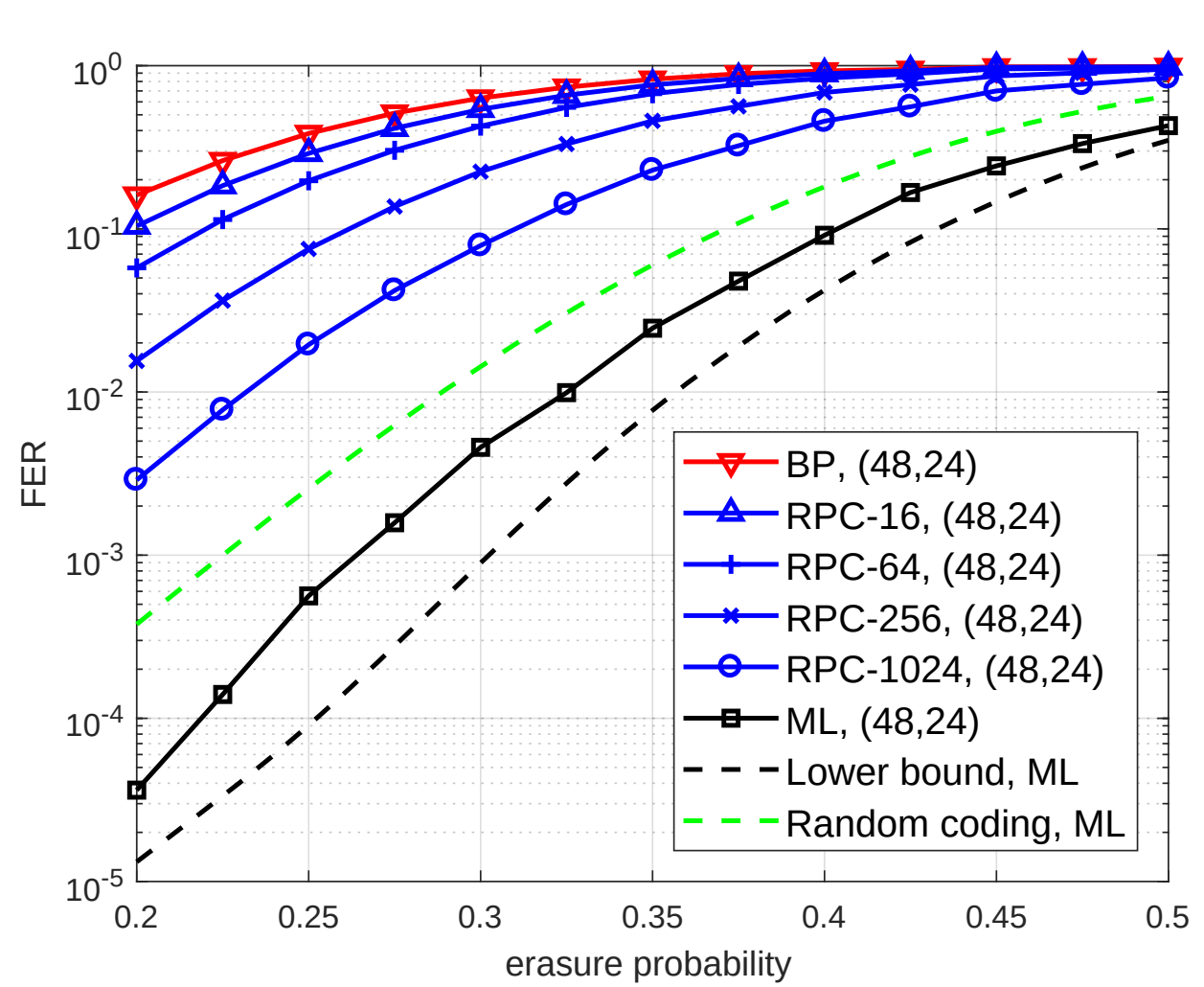

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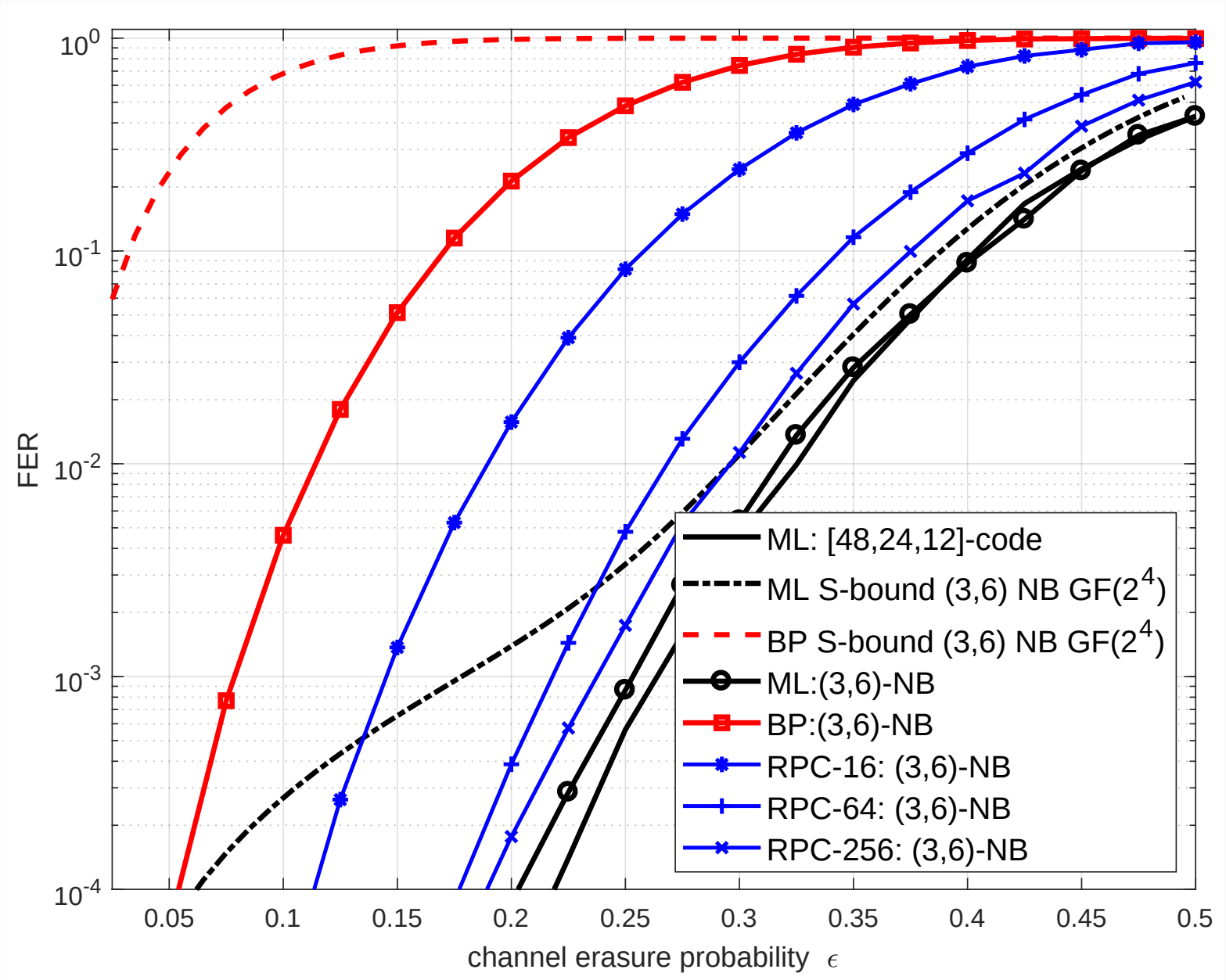
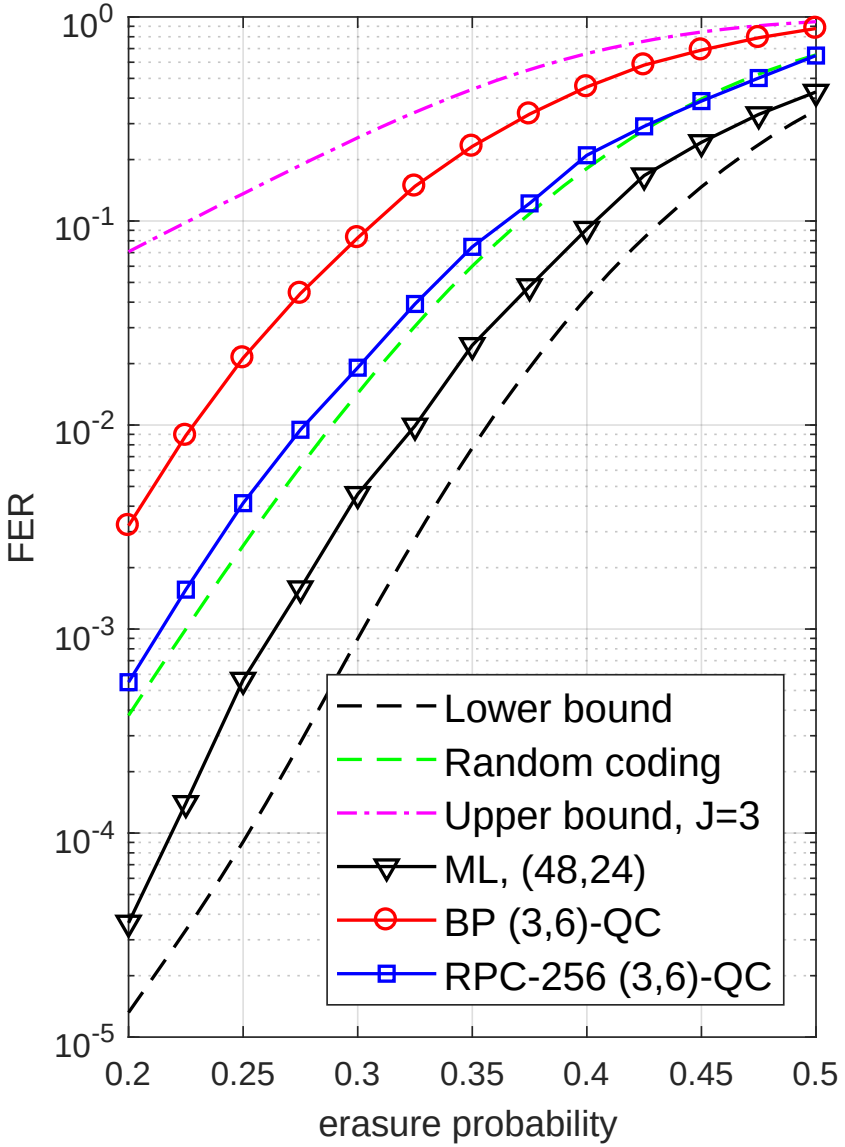
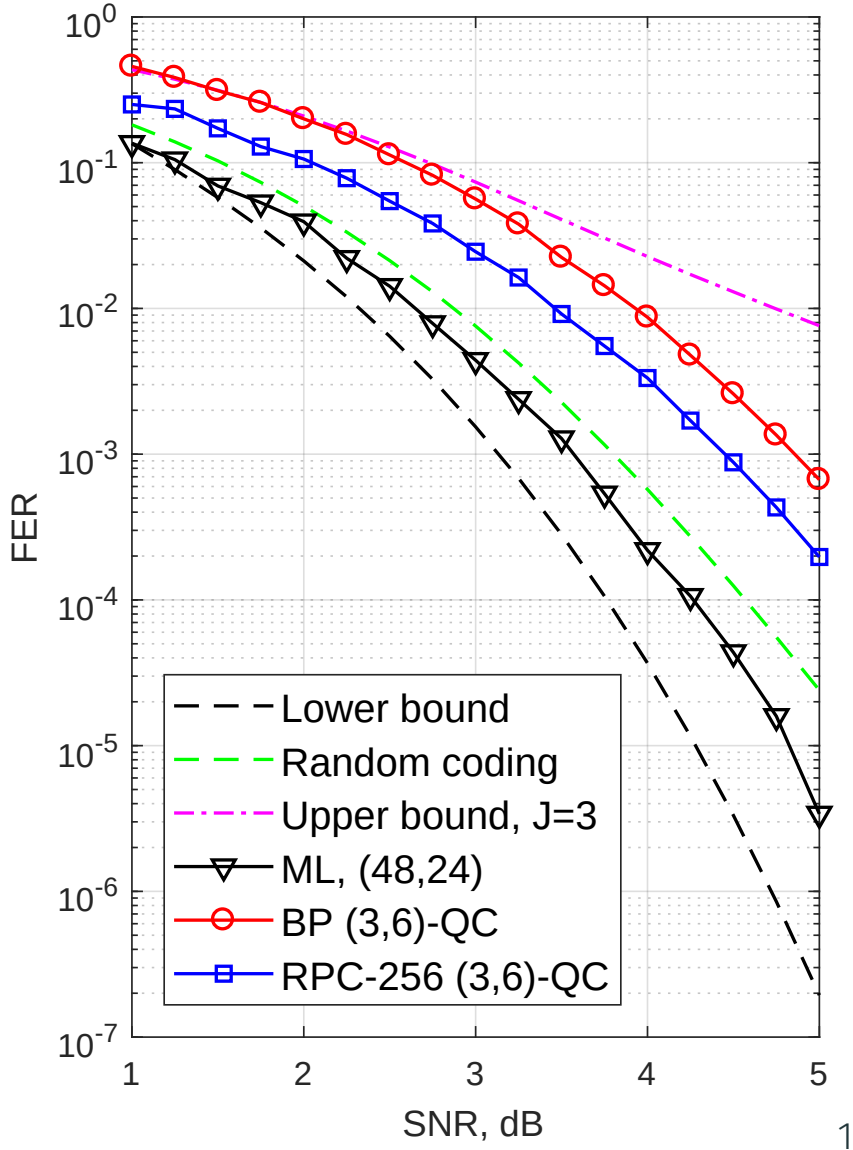


But what about BAWGN channel?

BEC vs BAWGNC





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- **Adding redundant rows works on BAWGNC too!**

Open problem

What code we want to contstruct for RPC:

- large d_{min}
- large d_{stop}
- small d_{dual}

Thank you

Just in case

Ensemble-Average Spectra

(J, K)-regular Gallager ensemble

Ensemble of (J, K)-regular parity-check matrices of LDPC $[n, k]$-codes

$$H = \begin{pmatrix} 1\,1\,1\,1\,1\,1 & & & \\ & 1\,1\,1\,1\,1\,1 & & \\ & & 1\,1\,1\,1\,1\,1 & \\ & & & 1\,1\,1\,1\,1\,1 \\ \hline \\ \hline \\ \end{pmatrix}$$

Each $M \times n$ strip is a permutation of columns of the first strip.

Weight-generating functions

$$G_n(s) = \sum_{w=0}^{n} A_{n,w} s^w$$

Recurrent coefficient calculation

Let

$$f(s) = \sum_{\ell \geq 0} f_\ell s^\ell$$

$$F_L(s) = (f(s))^L = \sum_{\ell \geq 0} F_{\ell,L} s^\ell$$

then

$$F_{\ell,L} = \begin{cases} f_\ell\,, & L = 1 \\ \sum_{i=0}^{\ell} f_i F_{\ell-i,L-1}\,, & L > 1 \end{cases}$$

Average weight spectrum

One row of H

$$g(s) = \sum_{i \text{ even}} \binom{K}{i} s^i = \frac{(1+s)^K + (1-s)^K}{2}$$

One strip of H

$$G(s) = \sum_{w=0}^{n} N_{n,w} s^w = (g(s))^M$$

Ensemble-average spectrum coefficients

$$E\{A_{n,w}\} = \binom{n}{w} (p(w))^J = \binom{n}{w}^{1-J} N_{n,w}^J$$

Other spectra

Stopping set spectrum

$$g(s) = \sum_{w=0,2,3,\ldots,K} \binom{K}{w} s^w = (1+s)^K - Ks$$

Weight spectrum

$$\phi(s) = \frac{1}{q-1} \sum_{w=1}^{m} \binom{m}{w} s^w = \frac{(1+s)^m - 1}{q-1}$$

$$g(\phi) = \frac{(1+(q-1)\phi)^K + (q-1)(1-\phi)^K}{q}$$

Can calculate fast!

Calculation of $d_{\min}$, d_{stop}

Minimum distance

$$\sum_{w=0}^{d_{\min}-1} A_{n,w} < 1$$

Stopping distance

$$\sum_{w=0}^{d_{\text{stop}}-1} B_{n,w} < 1$$

Numerical Results

Observations

- Random regular LDPC codes (esp. non-binary) with optimised J have minimum distance close to random linear codes

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- ...and gap decreases with decrease of R
- for LDPC, stopping distances are about half of min distances (for best LDPC – equal)

Example

Table 2: Examples of codes from the Gallager $(J, 2J)$ ensemble

(n, k, d)	J	$\hat{d}$	d_{GV}	d_{L}	d_{stop}	$\hat{d}_{\mathrm{stop}}$	ρ	$\hat{\rho}$
(40,24,6)	5	4	5	7	6	4	73	20
(60,35,8)	6	6	7	10	6	6	289	76
(90,49,10)	5	8	11	14	8	8	1118	86

Asymptotic Analysis

For binary images ($q = 2^m$)

Asymptotics for binary images of Gallager ensemble of nonbinary LDPC codes over $\mathbb{F}_q = \mathbb{F}_{2^m}$ (following Gallager)

$$E\{A_{n,w}\} \leq \binom{nm}{w}^{1-J} (g(\phi(s)))^{MJ} s^{-wJ}, \qquad \forall s$$

Replace s by e^ρ, find critical value $\delta = w/(mn)$, where

$$\lim_{n\to\infty} \frac{\ln E\{A_{nm,w}\}}{nm} = \min_\rho \left\{ (1-J) h_e(\delta) + \frac{J}{Km} \ln(g(\phi(e^\rho))) - \rho\delta J \right\} = 0$$

Normalised minimum distances

Table 3: Normalized minimum distances for binary and nonbinary LDPC code ensembles. Numbers in parentheses are typical asymptotic normalized stopping distances.

m	J				
	2	3	4	5	10
1	0	0.0227	0.0627	0.0843	0.1083
		(0.0180)	(0.0454)	(0.0580)	(0.0619)
2	0	0.0398	0.0774	0.0943	0.1095
3	0	0.0559	0.0880	0.1007	0.1099
4	0.0018	0.0686	0.0952	0.1046	0.1100
8	0.0396	0.0938	0.1066	0.1094	0.1100
16	0.0688	0.1049	0.1097	0.1100	0.1100