A Construction of Orbit Codes

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5ICMCTA, Vihula Manor. August 28-31, 2017

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1. Introduction
2. Preliminaries
3. Our construction
4. Open questions

Overview

1 Introduction

2 Preliminaries

3 Our construction

4 Open questions

Introduction

Network coding was introduced for satellite communication networks

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where the communication network is modeled as a finite directed acyclic multigraph, and information travels as packets from one or multiple sources to multiple receivers through intermediate nodes.

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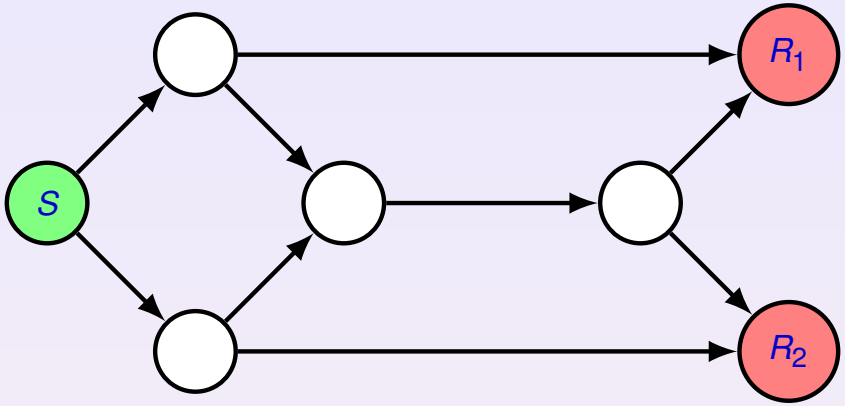

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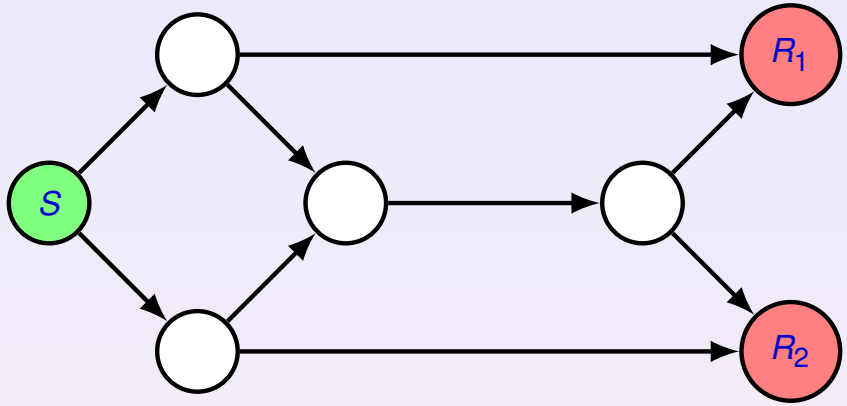
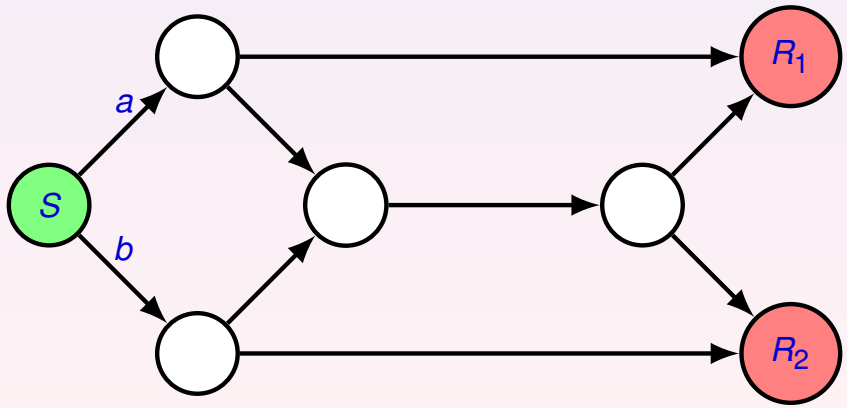


Assume that the source S wants to send the same information, a and b, to both receivers R_1 and R_2.



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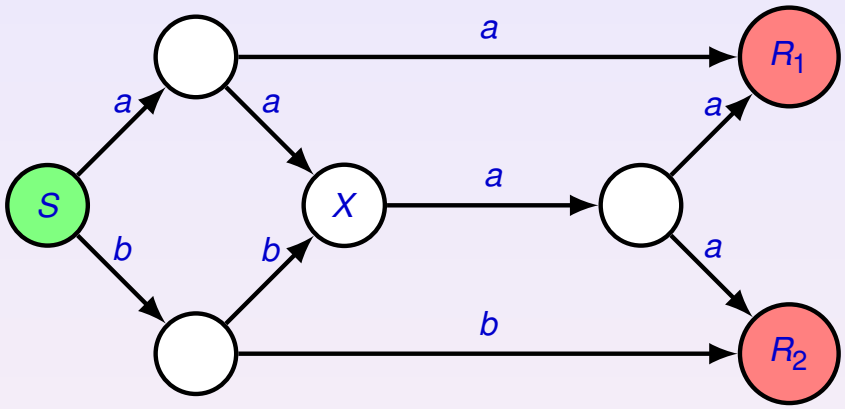
Under forwarding every inner node forwards the incoming information and thus has to decide on either a or b at the bottleneck vertex, marked by X.



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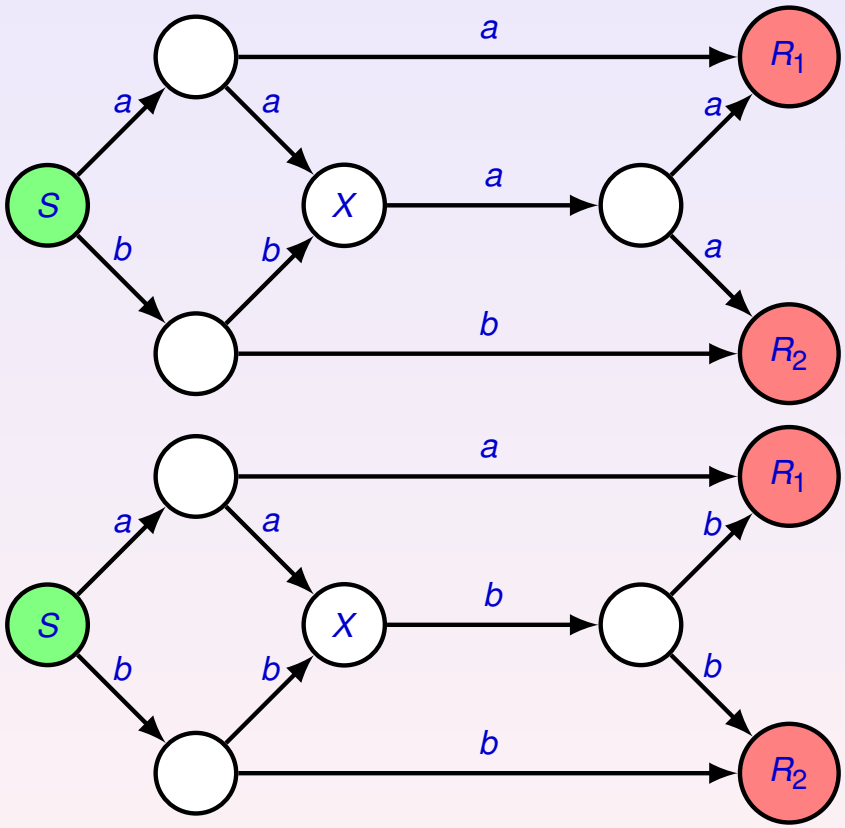

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Thus, R_2 does not receive a

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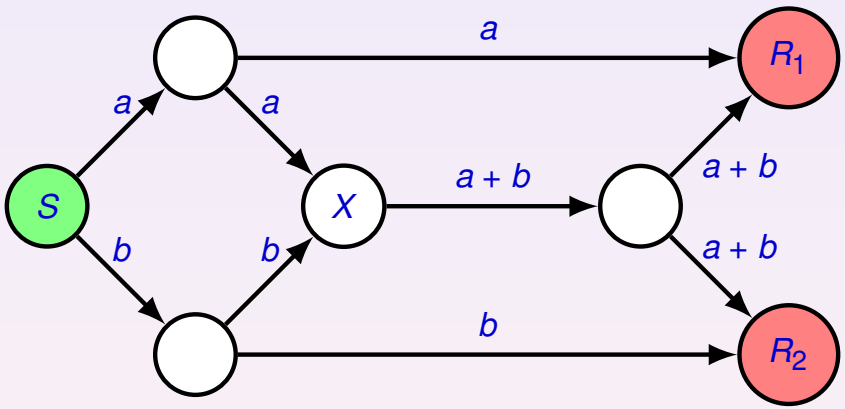
With linear network coding we allow the bottleneck vertex to send the sum of the two incoming informations, which allows both receivers to recover both a and b with a simple operation.



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In this linear network coding setting, when the topology of the underlying network is unknown or time-varying, one speaks of random (linear) network coding.

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An algebraic approach to network coding.
IEEE/ACM Transactions on Networking, **11(5)**: 782–795 (2003).

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Kötter and Kschischang proposed a mathematical description of network communications in which the transmitted messages are vector subspaces, rather than vectors.

R. KOETTER and F. R. KSCHISCHANG.
Coding for errors and erasures in random network coding.
IEEE Transactions on Information Theory, **54(8)**: 3579–3591 (2008).

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This approach is known as subspace codes.

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A.-L. Trautmann.

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Advances in Mathematics of Communications, **7(2)**: 147–160 (2013).

A.-L. Trautmann, F. Manganiello, M. Braun and J. Rosenthal.

Cyclic orbit codes.
IEEE Transactions on Information Theory, **59(11)**: 7386–7404 (2013).

A.-L. Trautmann, F. Manganiello and J. Rosenthal.

Orbit codes – a new concept in the area of network coding.
In *Proceedings of the 2010 IEEE Information Theory Workshop (ITW 2010)*. IEEE, Dublin, Ireland, August 2010.

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In the linear network coding setting, Trautmann, Manganiello and Rosenthal defined such codes as orbit codes by the action of subgroups of the general linear group.

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Orbit codes – a new concept in the area of network coding.
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Since then, several papers have been written about the structure of orbit codes.

F. BARDESTANI and A. IRANMANESH.
Cyclic orbit codes with the normalizer of a Singer subgroup.
Journal of Sciences, Islamic Republic of Iran, **26(1)**: 49–55 (2015).

A. GHATAK.
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These codes were introduced by Trautmann, Manganiello, Braun, and Rosenthal

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where the authors gave a characterization of a particular subclass of them.

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Specifically, we present a construction of an Abelian non-cyclic orbit code for which we calculate its cardinality and its minimum subspace distance.

Moreover, the minimum subspace distance of our code is maximal.

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2 Preliminaries

3 Our construction

4 Open questions

Preliminaries

Let $\mathbb{F}_q$ be the finite field of q elements, where q is a prime power.

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For any integer $n \geq 1$, the set $\mathcal{P}_q(n)$ of all vector subspaces of $\mathbb{F}_q^n$ forms a metric space with respect to the subspace distance defined by

$$d(\mathcal{U}, \mathcal{V}) = \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}), \quad \text{for all } \mathcal{U}, \mathcal{V} \in \mathcal{P}_q(n). \tag{1}$$

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For any integer k, where $0 \leq k \leq n$, the set $\mathcal{G}_q(k, n)$ of all k-dimensional vector subspaces of $\mathbb{F}_q^n$ is called the Grassmannian.

Obviously, $\mathcal{P}_q(n) = \bigcup_{k=0}^{n} \mathcal{G}_q(k, n)$.

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Otherwise, we can consider the complementary code (or dual code) of $\mathfrak{C}$:

$$\mathfrak{C}^{\perp} = \left\{ \mathcal{U}^{\perp} \subseteq \mathcal{G}_q(n-k, n) \mid \mathcal{U} \in \mathfrak{C} \right\}$$

obtained from the orthogonal subspaces of the codewords of $\mathfrak{C}$ which has the same length, cardinality and minimum distance.

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If $\mathcal{U} \in \mathcal{G}_q(k, n)$, then

$$\mathcal{U} = \text{rowspace}\,(U) = \left\{ \boldsymbol{x}U \mid \boldsymbol{x} \in \mathbb{F}_q^k \right\}$$

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If $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(k,n)$, then

$$d(\mathcal{U},\mathcal{V}) = 2(k - \dim(\mathcal{U}\cap\mathcal{V})) = 2\ \text{rank}\begin{bmatrix} U \\ V \end{bmatrix} - 2\,k \tag{2}$$

for some full-rank matrices $U, V \in \mathbb{F}_q^{k\times n}$ such that

$$\mathcal{U} = \text{rowspace}\,(U) \quad \text{and} \quad \mathcal{V} = \text{rowspace}\,(V)\,.$$

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We focus on constant dimension codes arising from group actions, which are called orbit codes

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$$\begin{array}{ccc} \mathcal{G}_q(k,n) \times \mathrm{GL}_n & \longrightarrow & \mathcal{G}_q(k,n) \\ (\mathcal{U}, A) & \mapsto & \mathcal{U} \cdot A \end{array}$$

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where $\mathcal{U} = \mathrm{rowspace}\,(U)$, $U \in \mathbb{F}_q^{k\times n}$ a full-rank matrix, and

$$\mathcal{U}\cdot A = \mathrm{rowspace}\,(UA)\,.$$

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Since, the subspace distance is GL_n-invariant:

$$d(\mathcal{U},\mathcal{V}) = d(\mathcal{U} \cdot A, \mathcal{V} \cdot A), \quad \text{for all } A \in \mathrm{GL}_n,$$

it follows that

$$d(\mathfrak{C}) = \min\{d(\mathcal{U}, \mathcal{U} \cdot A) \mid A \in \mathbf{G} \setminus \mathrm{stab}_{\mathbf{G}}(\mathcal{U})\}.$$

Preliminaries

Lemma 1

Assume that $A \in \mathbb{F}^{r\times r}$, $B \in \mathbb{F}^{s\times s}$, *and* $P \in \mathbb{F}^{r\times s}$ *and consider*

$$M = \begin{bmatrix} A & P \\ O & B \end{bmatrix}$$

where O *denotes the zero matrix of the appropriate size.*
If h *is a nonnegative integer, then*

$$M^h = \begin{bmatrix} A^h & \sigma_h(A,B,P) \\ O & B^h \end{bmatrix}$$

where

$$\sigma_h(A,B,P) = \begin{cases} O, & \text{if } h = 0, \\ \displaystyle\sum_{i=1}^{h} A^{h-i} P B^{i-1}, & \text{if } h \geq 1. \end{cases}$$

Overview

1 Introduction

2 Preliminaries

3 Our construction

4 Open questions

Our construction

Our aim is to construct an orbit code $\mathfrak{C}$ by considering the orbit of the vector subspace $\mathcal{U}_k$ generated by the first k unit vectors of $\mathbb{F}_q^n$, under the action of a concrete subgroup $\mathbf{H}$ of GL_n .

Our construction

Assume that

$$\mathbf{H} = \left\{ H \in \mathrm{GL}_n \mid H = \begin{bmatrix} H_{11} & H_{12} \\ O & H_{22} \end{bmatrix} \right.$$

$$\left. \text{where } H_{11} \in \mathrm{GL}_k \,,\, H_{12} \in \mathbb{F}_q^{k\times(n-k)},\, H_{22} \in \mathrm{GL}_{n-k} \right\}. \qquad (3)$$

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Since $\mathcal{U}_k = \text{rowspace}\left(\begin{bmatrix} I_k & O \end{bmatrix}\right)$, it follows that

$$\mathfrak{C} = \{\mathcal{U}_k \cdot H \mid H \in \mathbf{H}\} = \left\{\text{rowspace}\left(\begin{bmatrix} H_{11} & H_{12} \end{bmatrix}\right) \mid H \in \mathbf{H}\right\}. \qquad (4)$$

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Moreover, according to (2) and (3), for any $H \in \mathbf{H}$

$$d(\mathcal{U}_k, \mathcal{U}_k \cdot H) = 2\ \text{rank} \begin{bmatrix} I_k & O \\ H_{11} & H_{12} \end{bmatrix} - 2k = 2\ \text{rank}\,(H_{12}) \leq 2k. \qquad (5)$$

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On the other hand

$$\mathrm{stab}_{\mathrm{GL}_n}(\mathcal{U}_k) = \left\{ \begin{bmatrix} A & O \\ Q & B \end{bmatrix} \mid A \in \mathrm{GL}_k\,,\ Q \in \mathbb{F}_q^{(n-k)\times k} \text{ and } B \in \mathrm{GL}_{n-k} \right\}.$$

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Since $\mathrm{stab}_{\mathbf{H}}(\mathcal{U}_k) = \mathbf{H} \cap \mathrm{stab}_{\mathrm{GL}_n}(\mathcal{U}_k)$, and using the previous partition of matrices of $\mathbf{H}$ given by (3), then

$$\mathrm{stab}_{\mathbf{H}}(\mathcal{U}_k) = \left\{ \begin{bmatrix} H_{11} & H_{12} \\ O & H_{22} \end{bmatrix} \in \mathbf{H} \mid H_{12} = O \right\}. \tag{6}$$

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Let us denote by J_ℓ the Jordan block of size $\ell \times \ell$ associated to 1:

$$J_\ell = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

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Lemma 2

If $q^{t-1} < \ell \leq q^t$ for some nonnegative integer t, then $o(J_\ell) = q^t$.

Our construction

We consider the following two matrices of GL_n :

$$S = \begin{bmatrix} \lambda I_{n-k} & X \\ O & I_k \end{bmatrix}, \qquad T = \begin{bmatrix} J_{n-k} & Y \\ O & \lambda I_k \end{bmatrix} \quad \text{where } X, Y \in \mathbb{F}_q^{(n-k)\times k}. \tag{7}$$

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Lemma 3

For the matrices S and T given by (7)*, it follows that*

1. $o(S) = o(\lambda) = q - 1$.
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Lemma 4

1. $\langle S \rangle \cap \langle T \rangle = \{ I_n \}$.
2. *If $X \neq O$ and $Y = \frac{1}{1-\lambda} (\lambda I_{n-k} - J_{n-k})\, X$, then $ST = TS$.*

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$$\begin{aligned} \mathbf{H} &= \langle S, T \rangle = \langle S \rangle \times \langle T \rangle \\ &= \left\{ S^a T^b \mid 0 \leq a < q-1,\ 0 \leq b < q(q-1) \right\} \cong C_{q-1} \times C_{q(q-1)}, \end{aligned} \tag{8}$$

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that is, $\mathbf{H}$ is an Abelian non-cyclic group of order $q(q-1)^2$ and the elements of $\mathbf{H}$ present the following form

$$\begin{aligned} S^a T^b &= \begin{bmatrix} \lambda^a I_{n-k} & \sigma_a(\lambda I_{n-k}, I_k, X) \\ O & I_k \end{bmatrix} \begin{bmatrix} J_{n-k}^b & \sigma_b(J_{n-k}, \lambda I_k, Y) \\ O & \lambda^b I_k \end{bmatrix} \\ &= \begin{bmatrix} \lambda^a J_{n-k}^b & \lambda^a \sigma_b(J_{n-k}, \lambda I_k, Y) + \lambda^b \sigma_a(\lambda I_{n-k}, I_k, X) \\ O & \lambda^b I_k \end{bmatrix}. \end{aligned} \tag{9}$$

Our construction

Lemma 5

For any nonnegative integer b it follows that

$$J^b_{n-k} = \begin{bmatrix} 1 & \binom{b}{1} & \binom{b}{2} & \cdots & \binom{b}{n-k-2} & \binom{b}{n-k-1} \\ 0 & 1 & \binom{b}{1} & \cdots & \binom{b}{n-k-3} & \binom{b}{n-k-2} \\ 0 & 0 & 1 & \cdots & \binom{b}{n-k-4} & \binom{b}{n-k-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{b}{1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Our construction

Lemma 6

For matrices X and Y as in Lemma 4.2, then

$$\lambda^a \sigma_b(J_{n-k}, \lambda I_k, Y) + \lambda^b \sigma_a(\lambda I_{n-k}, I_k, X)$$
$$= \begin{bmatrix} c & e_1 & e_2 & \cdots & e_{n-k-2} & e_{n-k-1} \\ 0 & c & e_1 & \cdots & e_{n-k-3} & e_{n-k-2} \\ 0 & 0 & c & \cdots & e_{n-k-4} & e_{n-k-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & e_1 \\ 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix} X,$$

where $c = \dfrac{\lambda^a - \lambda^b}{\lambda - 1}$ *and* $e_i = \dfrac{\lambda^a}{\lambda - 1}\binom{b}{i}$, *for* $i = 1, 2, \ldots, n-k-1$.

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Lemma 7

With the notation of Lemma 6, $c = 0$ if, and only if, $a \equiv b \pmod{(q-1)}$.

Our construction

So, we consider the following partition

$$S^a T^b = \left[\begin{array}{c|c} \lambda^a J^b_{n-k} & \lambda^a \sigma_b(J_{n-k}, \lambda I_k, Y) + \lambda^b \sigma_a(\lambda I_{n-k}, I_k, X) \\ \hline O & \lambda^b I_k \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} & \overbrace{}^{n-2k} & \overbrace{}^{k} \\ \lambda^a J^b_k & (S^a T^b)_{12} & (S^a T^b)_{13} \\ \hline O & \lambda^a J^b_{n-2k} & (S^a T^b)_{23} \\ \hline O & O & \lambda^b I_k \end{array}\right] \begin{array}{l} \}k \\ \\ \}n-2k \\ \\ \end{array}$$

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$$= \left[\begin{array}{c|c|c} \lambda^a J^b_k & (S^a T^b)_{12} & (S^a T^b)_{13} \\ \hline O & \lambda^a J^b_{n-2k} & (S^a T^b)_{23} \\ \hline O & O & \lambda^b I_k \end{array}\right] \begin{array}{l} \}k \\ \\ \}n-2k \end{array}$$

($n-2k$ over the second column, k over the third column)

where, as a consequence of Lemma 5,

$$(S^a T^b)_{12} = \lambda^a \begin{bmatrix} \binom{b}{k} & \binom{b}{k+1} & \cdots & \binom{b}{n-k-1} \\ \binom{b}{k-1} & \binom{b}{k} & \cdots & \binom{b}{n-k-2} \\ \vdots & \vdots & & \vdots \\ \binom{b}{1} & \binom{b}{2} & \cdots & \binom{b}{n-2k} \end{bmatrix}. \qquad (10)$$

Our construction

Moreover, if we assume that

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{where } X_1 \in \mathbb{F}_q^{k\times k},\ X_2 \in \mathbb{F}_q^{(n-2k)\times k},$$

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Moreover, if we assume that

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then, from Lemma 6, it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} c & e_1 & e_2 & \cdots & e_{k-2} & e_{k-1} \\ 0 & c & e_1 & \cdots & e_{k-3} & e_{k-2} \\ 0 & 0 & c & \cdots & e_{k-4} & e_{k-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c & e_1 \\ 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix} X_1$$

$$+ \begin{bmatrix} e_k & e_{k+1} & e_{k+2} & \cdots & e_{n-k-2} & e_{n-k-1} \\ e_{k-1} & e_k & e_{k+1} & \cdots & e_{n-k-3} & e_{n-k-2} \\ e_{k-2} & e_{k-1} & e_k & \cdots & e_{n-k-4} & e_{n-k-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & e_3 & e_4 & \cdots & e_{n-2k-2} & e_{n-2k-1} \\ e_1 & e_2 & e_3 & \cdots & e_{n-2k-3} & e_{n-2k} \end{bmatrix} X_2. \qquad (11)$$

Our construction

According to (4), the orbit code $\mathfrak{C}$ generated by the action of $\mathbf{H}$ on $\mathcal{U}_k$ is

$$\begin{aligned}\mathfrak{C} &= \left\{\mathcal{U}_k \cdot (S^a T^b) \mid 0 \leq a < q-1,\ 0 \leq b < q(q-1)\right\} \\ &= \left\{\text{rowspace}\left(\left[\begin{array}{ccc} \lambda^a J_k^b & (S^a T^b)_{12} & (S^a T^b)_{13} \end{array}\right]\right)\right. \\ &\qquad \left. \mid 0 \leq a < q-1,\ 0 \leq b < q(q-1)\right\}.\end{aligned}$$

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Note that, according to (5),

$$d(\mathcal{U}_k, \mathcal{U}_k \cdot (S^a T^b)) = 2 \text{ rank}\left[\begin{array}{cc} (S^a T^b)_{12} & (S^a T^b)_{13} \end{array}\right] \leq 2k.$$

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Our next goal will be to choose X in such a way that

$$\operatorname{rank}\left[\begin{array}{cc} (S^a T^b)_{12} & (S^a T^b)_{13} \end{array}\right] = k, \quad \text{for all } S^a T^b \in \mathbf{H} \setminus \operatorname{stab}_{\mathbf{H}}(\mathcal{U}_k).$$

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Consequently, $d(\mathfrak{C}) = 2k$, and the minimum distance of our code will be maximal for a concrete choice of X.

Our construction

Theorem 8

If $X_1 \neq O$, *then* $\mathrm{stab}_{\mathbf{H}}(\mathcal{U}_k) = \langle ST^q \rangle$.

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Our construction

Theorem 9

If

$$X_1 = I_k \quad \text{and} \quad X_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{12}$$

then

$$\operatorname{rank} \begin{bmatrix} (S^a T^b)_{12} & (S^a T^b)_{13} \end{bmatrix} = k, \quad \text{for all } S^a T^b \in \mathbf{H} \setminus \operatorname{stab}_{\mathbf{H}}(\mathcal{U}_k).$$

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then

$$\operatorname{rank} \begin{bmatrix} (S^a T^b)_{12} & (S^a T^b)_{13} \end{bmatrix} = k, \quad \text{for all } S^a T^b \in \mathbf{H} \setminus \operatorname{stab}_{\mathbf{H}}(\mathcal{U}_k).$$

Let $S^a T^b \in \mathbf{H} \setminus \operatorname{stab}_{\mathbf{H}}(\mathcal{U}_k)$. From Theorem 8, $0 \leq a < q-1$, $0 \leq b < q(q-1)$ and $b \neq qa$.

Our construction

Theorem 9

If

$$X_1 = I_k \quad \text{and} \quad X_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{12}$$

then

$$\operatorname{rank} \begin{bmatrix} (S^a T^b)_{12} & (S^a T^b)_{13} \end{bmatrix} = k, \quad \text{for all } S^a T^b \in \mathbf{H} \setminus \operatorname{stab}_{\mathbf{H}}(\mathcal{U}_k).$$

Let $S^a T^b \in \mathbf{H} \setminus \operatorname{stab}_{\mathbf{H}}(\mathcal{U}_k)$. From Theorem 8, $0 \leq a < q-1$, $0 \leq b < q(q-1)$ and $b \neq qa$.

This means that $e_1 = \dfrac{\lambda^a}{\lambda - 1} b \neq 0$.

Our construction

Now, by (11) and (12), it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} c+e_k & e_1 & e_2 & e_3 & \cdots & e_{k-2} & e_{k-1} \\ e_{k-1} & c & e_1 & e_2 & \cdots & e_{k-3} & e_{k-2} \\ e_{k-2} & 0 & c & e_1 & \cdots & e_{k-4} & e_{k-3} \\ e_{k-3} & 0 & 0 & c & \cdots & e_{k-5} & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & 0 & 0 & 0 & \cdots & c & e_1 \\ e_1 & 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix}.$$

Our construction

Now, by (11) and (12), it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} 0+e_k & e_1 & e_2 & e_3 & \cdots & e_{k-2} & e_{k-1} \\ e_{k-1} & 0 & e_1 & e_2 & \cdots & e_{k-3} & e_{k-2} \\ e_{k-2} & 0 & 0 & e_1 & \cdots & e_{k-4} & e_{k-3} \\ e_{k-3} & 0 & 0 & 0 & \cdots & e_{k-5} & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & 0 & 0 & 0 & \cdots & 0 & e_1 \\ e_1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

If $c = 0$, then $\det(S^a T^b)_{13} = (-1)^k e_1^k \neq 0$ and $\operatorname{rank}(S^a T^b)_{13} = k$.

Our construction

Now, by (11) and (12), it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} c+e_k & e_1 & e_2 & e_3 & \cdots & e_{k-2} & e_{k-1} \\ e_{k-1} & c & e_1 & e_2 & \cdots & e_{k-3} & e_{k-2} \\ e_{k-2} & 0 & c & e_1 & \cdots & e_{k-4} & e_{k-3} \\ e_{k-3} & 0 & 0 & c & \cdots & e_{k-5} & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & 0 & 0 & 0 & \cdots & c & e_1 \\ e_1 & 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix}.$$

If $c = 0$, then $\det(S^a T^b)_{13} = (-1)^k e_1^k \neq 0$ and $\operatorname{rank}(S^a T^b)_{13} = k$.

If $c \neq 0$, it follows that $k - 1 \leq \operatorname{rank}(S^a T^b)_{13} \leq k$.

Our construction

Now, by (11) and (12), it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} c+e_k & e_1 & e_2 & e_3 & \cdots & e_{k-2} & e_{k-1} \\ e_{k-1} & c & e_1 & e_2 & \cdots & e_{k-3} & e_{k-2} \\ e_{k-2} & 0 & c & e_1 & \cdots & e_{k-4} & e_{k-3} \\ e_{k-3} & 0 & 0 & c & \cdots & e_{k-5} & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & 0 & 0 & 0 & \cdots & c & e_1 \\ e_1 & 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix}.$$

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If $c \neq 0$, it follows that $k - 1 \leq \operatorname{rank}(S^a T^b)_{13} \leq k$.

Assume that $\operatorname{rank}(S^a T^b)_{13} = k - 1$.

Our construction

Now, by (11) and (12), it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} c+e_k & e_1 & e_2 & e_3 & \cdots & e_{k-2} & e_{k-1} \\ e_{k-1} & c & e_1 & e_2 & \cdots & e_{k-3} & e_{k-2} \\ e_{k-2} & 0 & c & e_1 & \cdots & e_{k-4} & e_{k-3} \\ e_{k-3} & 0 & 0 & c & \cdots & e_{k-5} & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & 0 & 0 & 0 & \cdots & c & e_1 \\ e_1 & 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix}.$$

If $c = 0$, then $\det(S^a T^b)_{13} = (-1)^k e_1^k \neq 0$ and $\operatorname{rank}(S^a T^b)_{13} = k$.

If $c \neq 0$, it follows that $k - 1 \leq \operatorname{rank}(S^a T^b)_{13} \leq k$.

Assume that $\operatorname{rank}(S^a T^b)_{13} = k - 1$. Then

$$0 = \det(S^a T^b)_{13}$$

Our construction

Now, by (11) and (12), it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} c+e_k & e_1 & e_2 & e_3 & \cdots & e_{k-2} & e_{k-1} \\ e_{k-1} & c & e_1 & e_2 & \cdots & e_{k-3} & e_{k-2} \\ e_{k-2} & 0 & c & e_1 & \cdots & e_{k-4} & e_{k-3} \\ e_{k-3} & 0 & 0 & c & \cdots & e_{k-5} & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & 0 & 0 & 0 & \cdots & c & e_1 \\ e_1 & 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix}.$$

If $c = 0$, then $\det(S^a T^b)_{13} = (-1)^k e_1^k \neq 0$ and $\operatorname{rank}(S^a T^b)_{13} = k$.

If $c \neq 0$, it follows that $k - 1 \leq \operatorname{rank}(S^a T^b)_{13} \leq k$.

Assume that $\operatorname{rank}(S^a T^b)_{13} = k - 1$. Then

$$0 = \det(S^a T^b)_{13} = \det \begin{bmatrix} c+e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{bmatrix} \tag{13}$$

Our construction

Now, by (11) and (12), it follows that

$$(S^a T^b)_{13} = \begin{bmatrix} c+e_k & e_1 & e_2 & e_3 & \cdots & e_{k-2} & e_{k-1} \\ e_{k-1} & c & e_1 & e_2 & \cdots & e_{k-3} & e_{k-2} \\ e_{k-2} & 0 & c & e_1 & \cdots & e_{k-4} & e_{k-3} \\ e_{k-3} & 0 & 0 & c & \cdots & e_{k-5} & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ e_2 & 0 & 0 & 0 & \cdots & c & e_1 \\ e_1 & 0 & 0 & 0 & \cdots & 0 & c \end{bmatrix}.$$

If $c = 0$, then $\det(S^a T^b)_{13} = (-1)^k e_1^k \neq 0$ and $\operatorname{rank}(S^a T^b)_{13} = k$.

If $c \neq 0$, it follows that $k - 1 \leq \operatorname{rank}(S^a T^b)_{13} \leq k$.

Assume that $\operatorname{rank}(S^a T^b)_{13} = k - 1$. Then

$$0 = \det(S^a T^b)_{13} = \det \begin{bmatrix} c+e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{bmatrix} \tag{13}$$

where $\boldsymbol{u} = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_{k-2} \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} e_{k-1} & e_{k-2} & e_{k-3} & \cdots & e_1 \end{bmatrix}$, and $\det(M) = c^{k-1} \neq 0$.

Our construction

Assume that $\operatorname{rank}\left[\begin{array}{cc}(S^aT^b)_{12} & (S^aT^b)_{13}\end{array}\right] = k-1$.

Our construction

Assume that $\operatorname{rank}\left[\begin{array}{cc}(S^aT^b)_{12} & (S^aT^b)_{13}\end{array}\right] = k-1$. Then

$$k-1 = \operatorname{rank}\left[\begin{array}{c|ccccc} \lambda^a\binom{b}{k} & e_1 & e_2 & e_3 & \cdots & e_{k-1} \\ \hline \lambda^a\binom{b}{k-1} & c & e_1 & e_2 & \cdots & e_{k-2} \\ \lambda^a\binom{b}{k-2} & 0 & c & e_1 & \cdots & e_{k-3} \\ \lambda^a\binom{b}{k-3} & 0 & 0 & c & \cdots & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda^a\binom{b}{1} & 0 & 0 & 0 & \cdots & c \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{array}\right], \qquad (14)$$

where we have used that $\lambda^a\binom{b}{i} = (\lambda-1)e_i$.

Our construction

Assume that $\operatorname{rank}\left[\begin{array}{cc}(S^aT^b)_{12} & (S^aT^b)_{13}\end{array}\right] = k-1$. Then

$$k-1 = \operatorname{rank}\left[\begin{array}{c|cccc} \lambda^a\binom{b}{k} & e_1 & e_2 & e_3 & \cdots & e_{k-1} \\ \hline \lambda^a\binom{b}{k-1} & c & e_1 & e_2 & \cdots & e_{k-2} \\ \lambda^a\binom{b}{k-2} & 0 & c & e_1 & \cdots & e_{k-3} \\ \lambda^a\binom{b}{k-3} & 0 & 0 & c & \cdots & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda^a\binom{b}{1} & 0 & 0 & 0 & \cdots & c \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{array}\right], \qquad (14)$$

where we have used that $\lambda^a\binom{b}{i} = (\lambda-1)e_i$.

Now,

$$0 = \det\left[\begin{array}{cc} c+e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{array}\right] = \det\left[\begin{array}{cc} c & \boldsymbol{u} \\ \boldsymbol{0}^T & M \end{array}\right] + \det\left[\begin{array}{cc} e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{array}\right] = c^k + 0,$$

which is a contradiction since $c \neq 0$.

Our construction

Assume that $\operatorname{rank}\left[\begin{array}{cc}(S^aT^b)_{12} & (S^aT^b)_{13}\end{array}\right] = k-1$. Then

$$k-1 = \operatorname{rank}\left[\begin{array}{c|cccc} \lambda^a\binom{b}{k} & e_1 & e_2 & e_3 & \cdots & e_{k-1} \\ \hline \lambda^a\binom{b}{k-1} & c & e_1 & e_2 & \cdots & e_{k-2} \\ \lambda^a\binom{b}{k-2} & 0 & c & e_1 & \cdots & e_{k-3} \\ \lambda^a\binom{b}{k-3} & 0 & 0 & c & \cdots & e_{k-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda^a\binom{b}{1} & 0 & 0 & 0 & \cdots & c \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{array}\right], \qquad (14)$$

where we have used that $\lambda^a\binom{b}{i} = (\lambda-1)e_i$.

Now,

$$0 = \det\left[\begin{array}{cc} c+e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{array}\right] = \det\left[\begin{array}{cc} c & \boldsymbol{u} \\ \boldsymbol{0}^T & M \end{array}\right] + \det\left[\begin{array}{cc} e_k & \boldsymbol{u} \\ \boldsymbol{v}^T & M \end{array}\right] = c^k + 0,$$

which is a contradiction since $c \neq 0$.

Consequently $\operatorname{rank}\left[\begin{array}{cc}(S^aT^b)_{12} & (S^aT^b)_{13}\end{array}\right] = k$, for all $S^aT^b \in \mathbf{H} \setminus \operatorname{stab}_{\mathbf{H}}(\mathcal{U}_k)$.

Our construction

Our construction

Theorem 10

Assume that q is a prime number such that $q \geq n - k$ where k and n are nonnegative integers such that $2k \leq n$.
Assume also that $\lambda \in \mathbb{F}_q$ is a primitive element.
Consider the upper triangular matrices S and T defined in (7), and
$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ *where X_1 and X_2 are defined in (12).*
If $\mathbf{H} = \langle S, T \rangle$ is the Abelian non-cyclic subgroup of GL_n given in (8), and $\mathfrak{C}$ is the orbit code defined in (4), then $|\mathfrak{C}| = q(q-1)$ and $d(\mathfrak{C}) = 2k$.

Overview

1 Introduction

2 Preliminaries

3 Our construction

4 Open questions

Open questions

If q is a prime number such that $n - k \leq q$, then

- $e_1 = 0$ if and only if $b = qa$ for $0 \leq a < q - 1$,
- $e_1 \neq 0$, for the remaining cases.

Open questions

If q is a prime number such that $n - k \leq q$, then

- $e_1 = 0$ if and only if $b = qa$ for $0 \leq a < q - 1$,
- $e_1 \neq 0$, for the remaining cases.

$$\mathrm{stab}_{\mathbf{H}}(\mathcal{U}_k) = \langle ST^q \rangle = \left\{ S^a T^b \in \mathbf{H} \mid b = qa,\ 0 \leq a < q - 1 \right\} \cong C_{q-1}.$$

$$(S^a T^b)_{12} = O, \qquad (S^a T^b)_{13} = O.$$

Open questions

If q is a prime number such that $n-k \leq q$, then

- $e_1 = 0$ if and only if $b = qa$ for $0 \leq a < q-1$,
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$$(S^a T^b)_{12} = O, \qquad (S^a T^b)_{13} = O.$$

However, if $q = p^r$ where p is a prime number and $r \geq 2$, then

- $e_1 = 0$ if and only if $b = pa$ for $0 \leq a < q-1$,
- $e_1 \neq 0$, for the remaining cases.

Open questions

If q is a prime number such that $n-k \leq q$, then

- $e_1 = 0$ if and only if $b = qa$ for $0 \leq a < q-1$,
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- $e_1 = 0$ if and only if $b = pa$ for $0 \leq a < q-1$,
- $e_1 \neq 0$, for the remaining cases.

In this case

$$\mathrm{stab}_{\mathbf{H}}(\mathcal{U}_k) = \begin{cases} \langle ST^p \rangle, & \text{if } n-k \leq p, \\ ?, & \text{if } p < n-k \leq q. \end{cases}$$

Open questions

If q is a prime number such that $n-k \leq q$, then

- $e_1 = 0$ if and only if $b = qa$ for $0 \leq a < q-1$,
- $e_1 \neq 0$, for the remaining cases.

$$\mathrm{stab}_{\mathbf{H}}(\mathcal{U}_k) = \langle ST^q \rangle = \left\{ S^a T^b \in \mathbf{H} \mid b = qa,\ 0 \leq a < q-1 \right\} \cong C_{q-1}.$$

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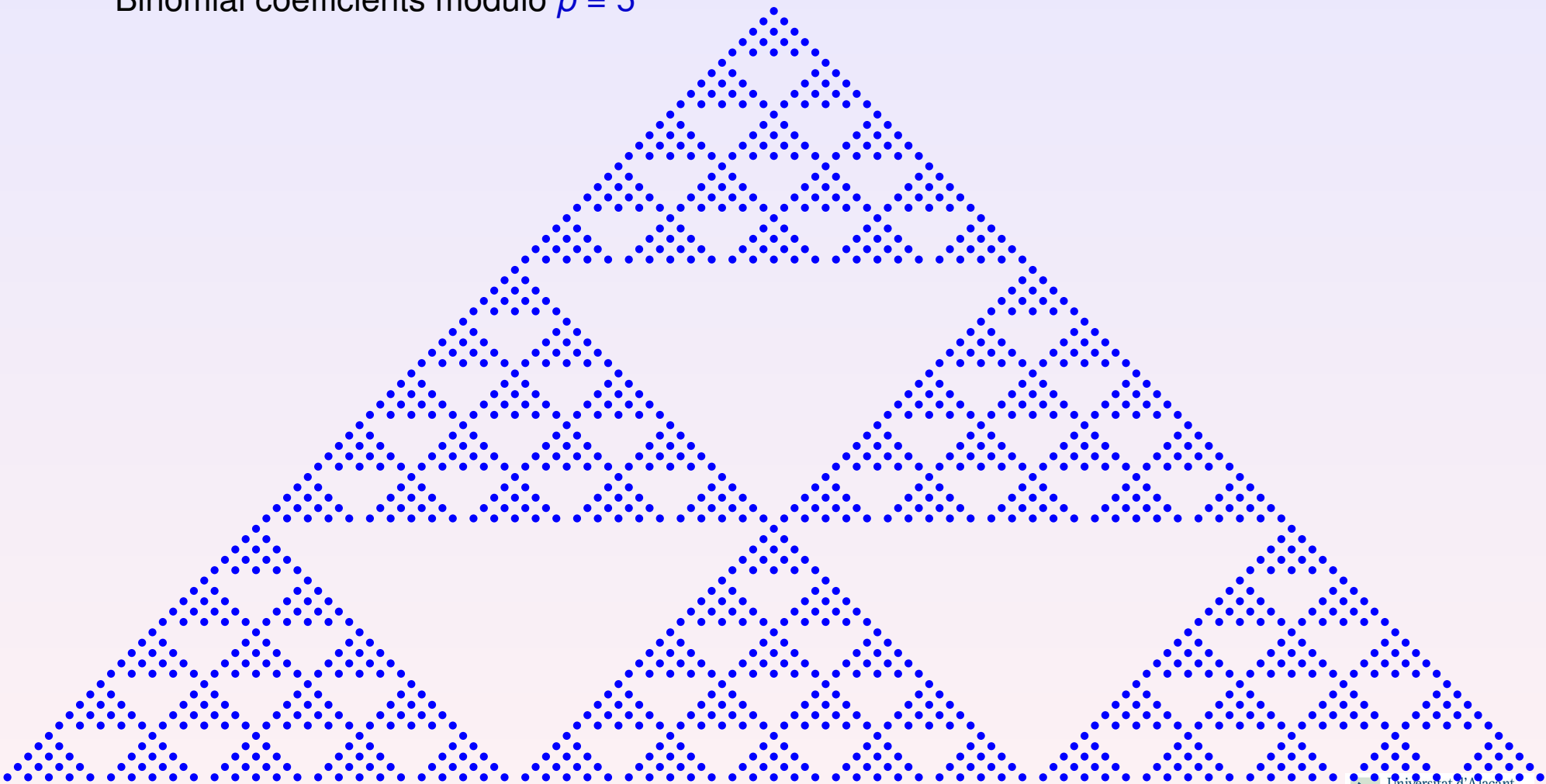
In this case

$$\mathrm{stab}_{\mathbf{H}}(\mathcal{U}_k) = \begin{cases} \langle ST^p \rangle, & \text{if } n-k \leq p, \\ ?, & \text{if } p < n-k \leq q. \end{cases}$$

$$(S^a T^b)_{12} = \lambda^a \begin{bmatrix} \binom{b}{k} & \binom{b}{k+1} & \cdots & \binom{b}{n-k-1} \\ \binom{b}{k-1} & \binom{b}{k} & \cdots & \binom{b}{n-k-2} \\ \vdots & \vdots & & \vdots \\ \binom{b}{1} & \binom{b}{2} & \cdots & \binom{b}{n-2k} \end{bmatrix} \neq O.$$

Binomial coefficients modulo $p = 5$



A Construction of Orbit Codes

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Overview

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