Analysis of two tracing traitor schemes via coding theory

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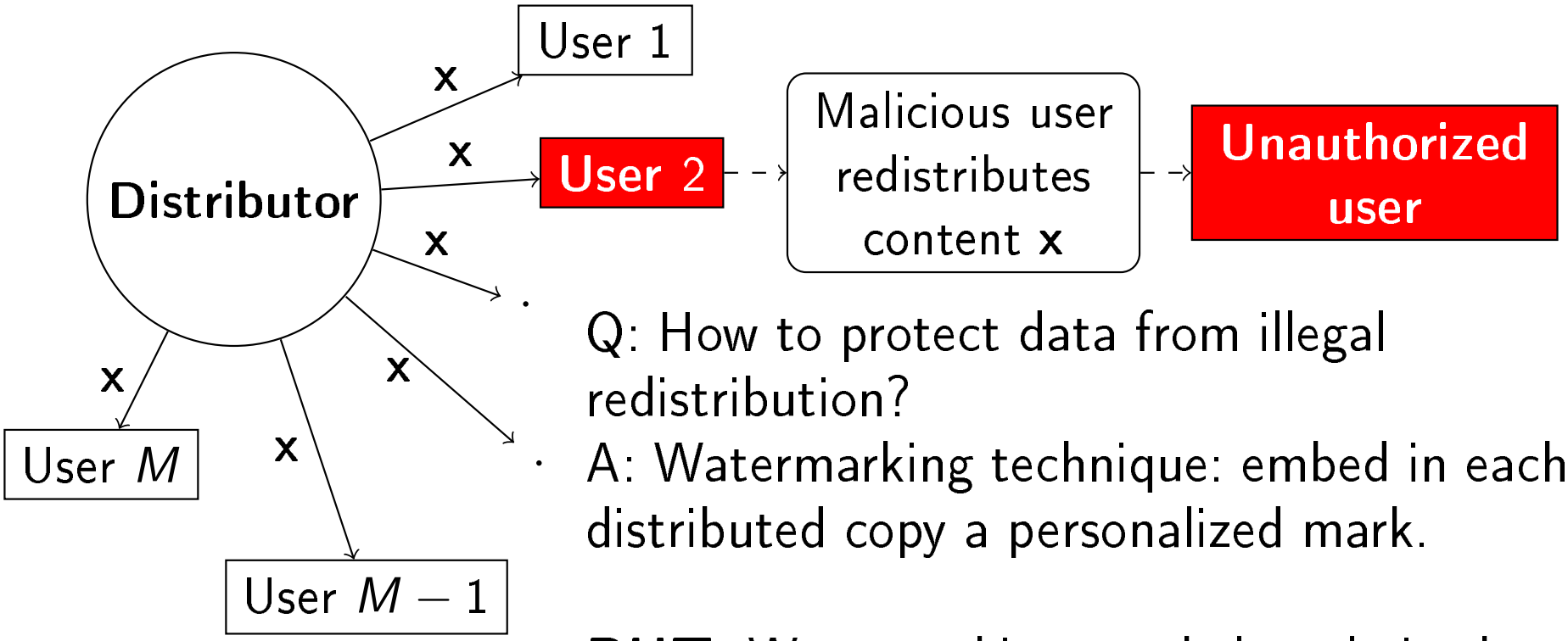
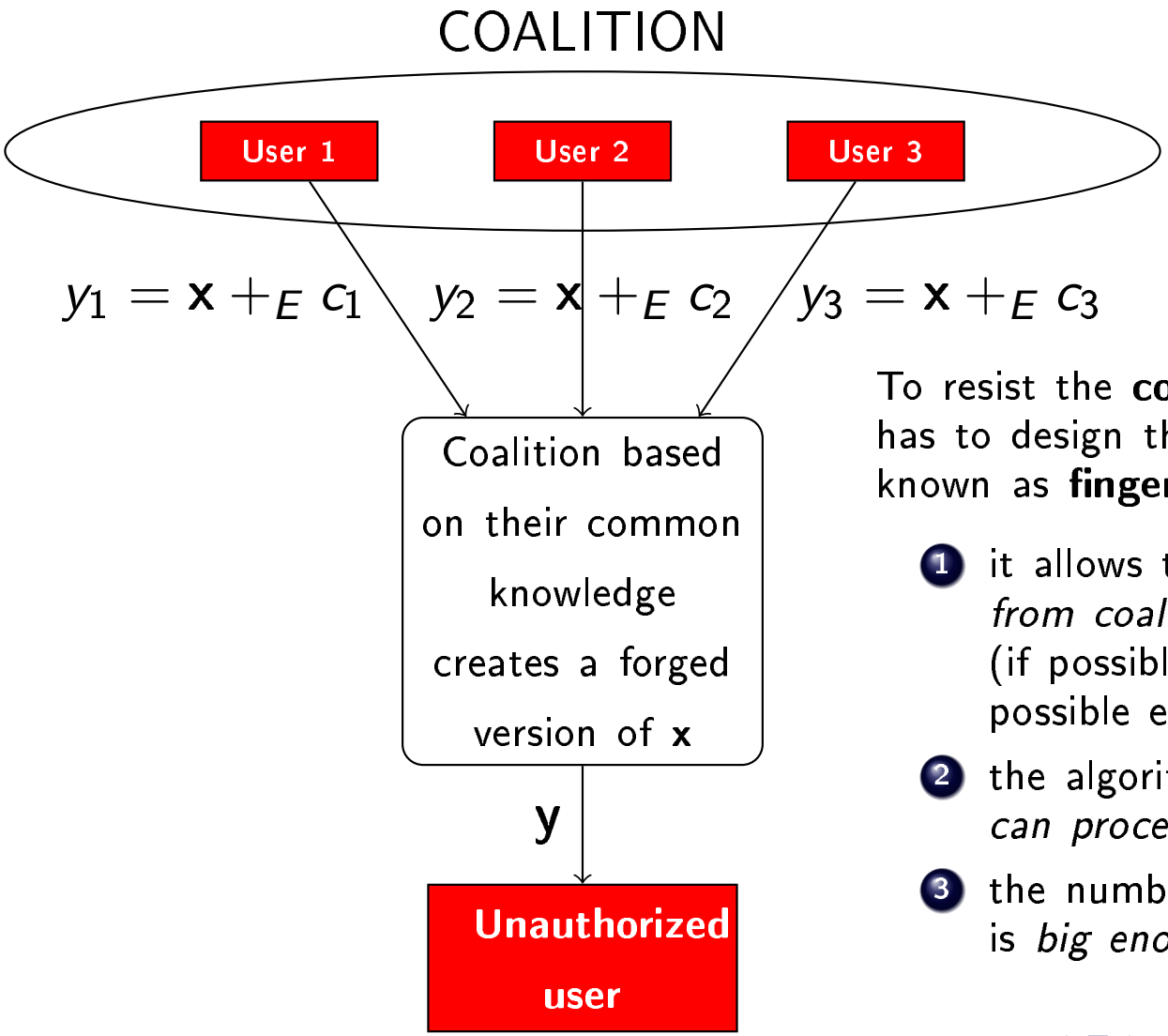
Watermarking vs Collusion attack



Q: How to protect data from illegal redistribution?
A: Watermarking technique: embed in each distributed copy a personalized mark.

BUT: Watermarking can help only in the case of **single** traitor.

Fingerprinting codes against collusion attacks



To resist the **collusion attack** distributor has to design the set of user marks, known as **fingerprinting code**, such that

1. it allows to identify *at least one traitor from coalition* with zero error (if possible) or with the minimum possible error rate,
2. the algorithm of identification *can proceed in real time*,
3. the number of authorized users is *big enough*.

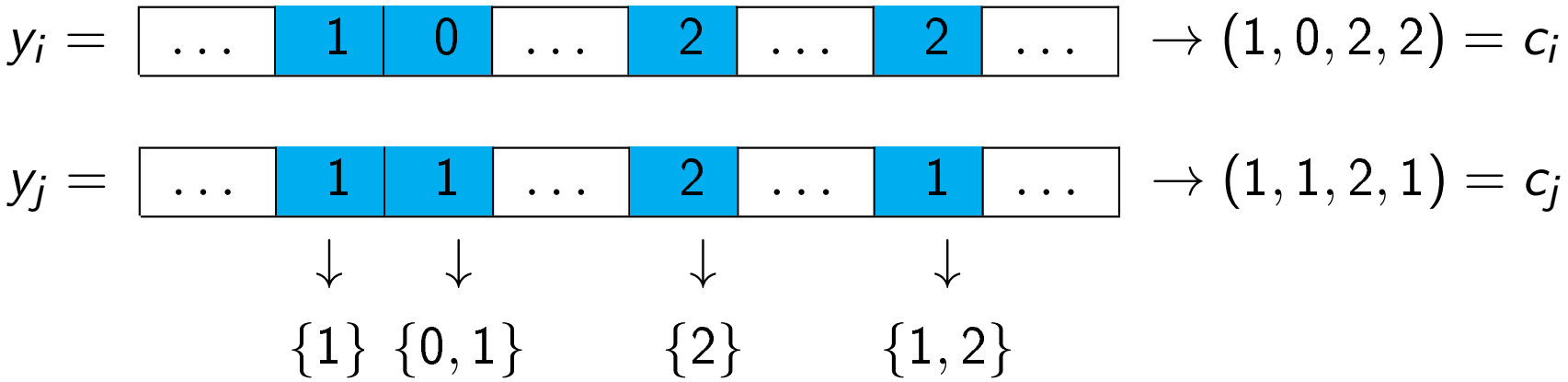
Collusion attack paradigm

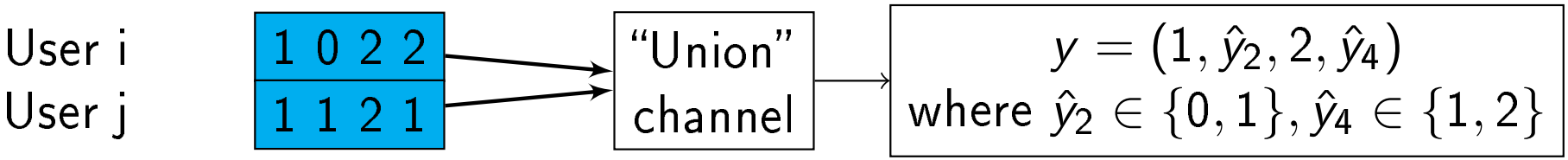
Let $C = \{c_1, ..., c_M\} \subset A^n$ be a fingerprinting code. All users know the code.
We assume that no more than t users can form a coalition $U = \{i_1, ..., i_k\}, k \leq t$.
Let $< U >$ be a set of all forged vectors that coalition U can generate according to specificity of the considered model (examples are on the next slide).
Goal of the distributor: construct a code C that satisfies the t-*Identifiable parent property* (t-IPP for short).
Definition. A code C has t-IPP property if for any vector $y \in A^n$

$$\bigcap_{U:\, |U| \leq t,\, y \in \langle U \rangle} U \neq \emptyset$$

or no one coalition U of cardinality t can create y.

Examples 1: t-IPP codes





Goal: for any t-coalition and any given y generated by the coalition the distributor can correctly identify at least one member of the coalition.

t-IPP codes

Formally, let $P_j(U) = \{u_j \mid u = (u_1, ..., u_n) \in U\}$, then

$$y \in \langle U \rangle_t := \{(y_1, ..., y_n) \in Q^n \mid y_j \in P_j(U)\}$$

Definition [Hollmann et al, 1998] A code C called t-**IPP code** if for any vector $y \in Q^n$

$$\bigcap_{U:\, |U| \le t,\, y \in \langle U \rangle_t} U \neq \emptyset$$

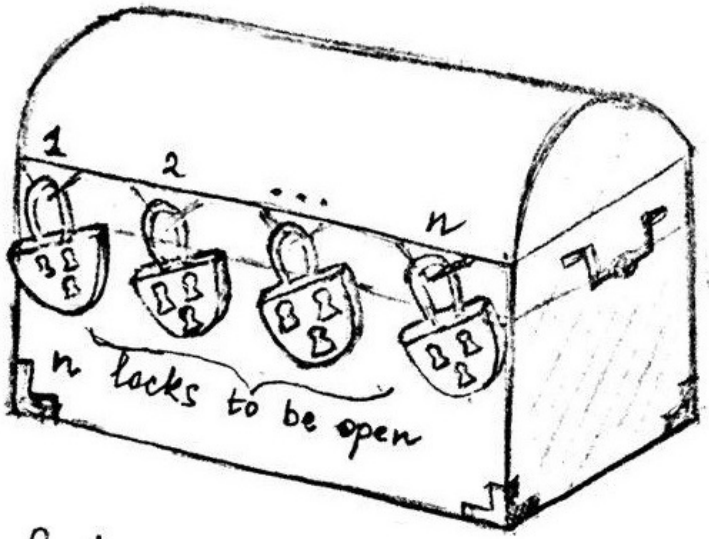
or no one coalition U of cardinality t can create y.

Let $R(C) = \frac{\log_q N}{n}$ — rate of a code C.

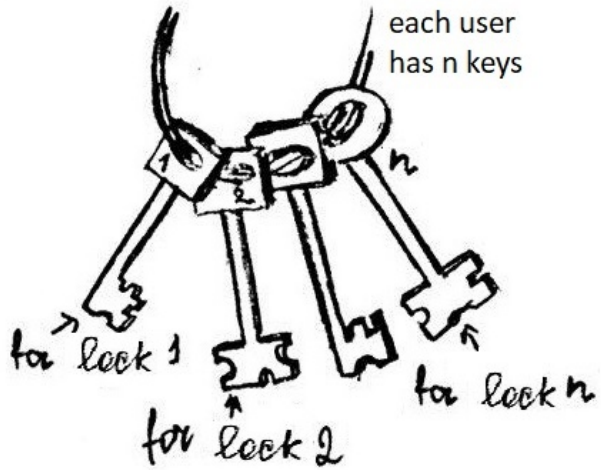
Theorem [Barg A. et al., 2001]
Good t-IPP code exists, i.e. $R \geq c(t) > 0 \Leftrightarrow t < q = |Q|$.

$\Rightarrow$ for the most interesting case of binary codes the IPP property does not hold even in the case of only two traitors.

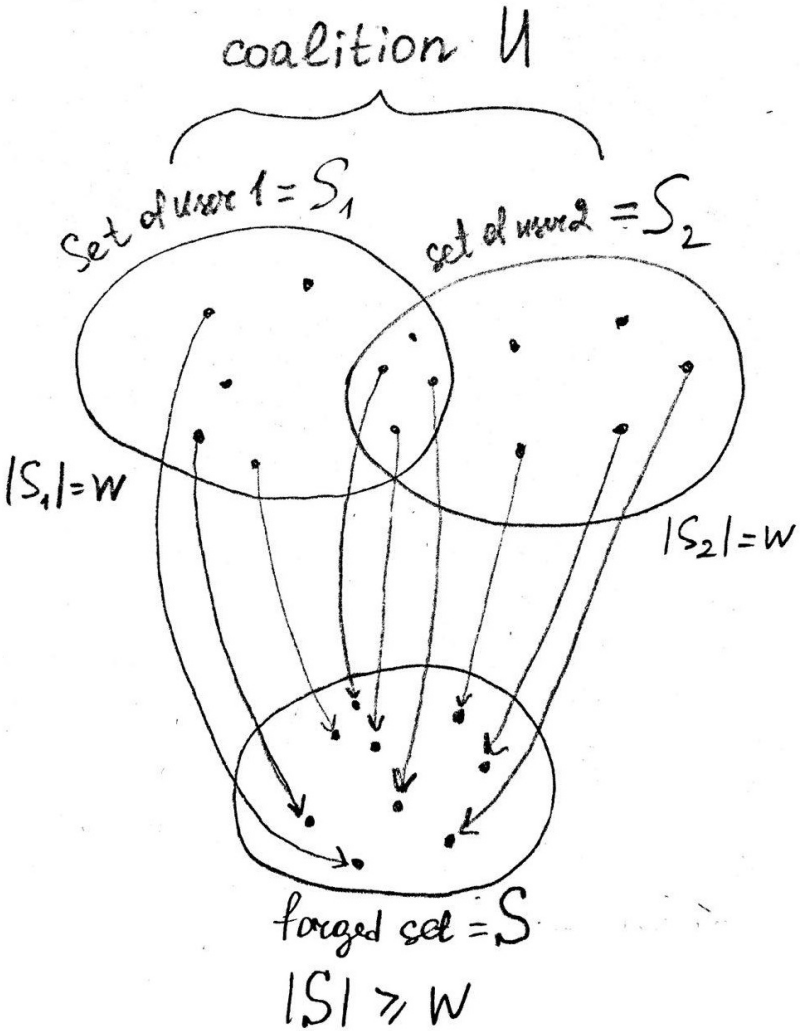
Toy example of IPP codes



each lock has q keyholes



Example 2: t-IPP set systems



Distributors chooses: a set of base keys X, $|X| = N$ and a parameter $w < N$.

i-th user has: the corresponding set of shares $S_i \subset X$, $|S_i| = w$.

Collusion attack: coalition $U = \{i_1, ..., i_k\}$ generates any element of

$$\langle U \rangle = \{S \mid S \subset \bigcup_{u \in U} S_u, |S| \geq w\}$$

t-IPP set systems

The goal: is, given any forged $\geq w-$subset of X, to be able to identify at least one user from the malicious coalition.

Definition

The family $F = \{S_1, ..., S_M\}$ of w-subsets of X, $|X| = N$ is called a t-IPP set system (t-IPPSS) if for every $S \subseteq X$, $|S| \geq w$ the intersection of all coalitions of size $\leq t$ that can generate S is either not empty, i.e.

$$\bigcap_{U:|U|\leq t, S\in<U>} U \neq \emptyset$$

or there is no $\leq t$-coalition that can generate S.

Remark: q-ary IPP codes of length n can be considered as a "weak" IPP set systems with $w = n$ and $N = qn$.

Traceability

Definition

Family $F = \{S_1, ..., S_M\}$ of w-subsets of X, $|X| = N$ is called t-**traceability set system** (t-TSS) if for any coalition $U = \{i_1, ..., i_k\} \subset [M]$, $k \leq t$ and any $S \in < U >$ holds

$$|S \cap S_j| < \max_{u \in U} |S \cap S_u| \text{ for any j} \notin U$$

It means that the search of malicious users reduces to the search of "closest" sets. For the corresponding binary codes, it's equivalent to the minimal distance decoding.
Moreover, any t-TSS is also t-IPPSS, so the lower bounds on the size of t-TSS are also valid for t-IPPSS.

Sufficient condition of t-traceability

Lemma

If $|S_i \cap S_j| < w/t^2$ for any $S_i, S_j \in F, i \neq j$, then F is a t-traceability set system.

Proof

Consider any coalition $U = \{i_1, ..., i_k\}$, $k \leq t$ and any $S \in < U >$.
Then, $\max_{i_l \in U} |S_{i_l} \cap S| \geq w/t$ since $|S| \geq w$.
On the other hand, for any $j \notin U$,

$$|S_j \cap S| \leq |S_j \cap (S_{i_1} \cup ... \cup S_{i_k})| \leq |S_j \cap S_{i_1}| + ... + |S_j \cap S_{i_k}| < t \cdot \frac{w}{t^2} = \frac{w}{t}, \text{ Q.E.D.}$$

The system of sets given in this lemma is the same as a constant-weight code of length N, weight w and distance $d \geq 2\tau n$ where $\tau = \omega(1 - 1/t^2)$ and $\omega := w/N$ (relative weight).

Let $M(N, \omega)$ be the maximum possible size of t-TSS with $|X| = N$ and weight ωN. Denote by

$$R_t^{TSS}(N, \omega) := \frac{\log_2 M(N, \omega)}{N}$$

the maximum possible rate and by

$$R_t(\omega) := \liminf_{N \to \infty} R_t^{TSS}(N, \omega)$$

.

Theorem

There exists a t-TSS code with rate

$$R_t(\omega) \geq H(\omega) - \omega H\left(\frac{\tau}{\omega}\right) - (1 - \omega) H\left(\frac{\tau}{1 - \omega}\right)$$

where H is binary entropy function and $\tau = \omega(1 - 1/t^2)$.

Proof of the theorem

Proof: Let $m_w(N, 2v)$ be the maximal cardinality of constant-weight code of length N, weight w and minimal distance $2v$. The corresponding Gilbert bound has the following form

$$m_w(N, 2v) \geq \frac{\binom{N}{w}}{\sum_{i=0}^{v-1} \binom{w}{i}\binom{N-w}{i}}$$

Asymptotically:

$$R = \frac{\log_2 m_w(N, 2v)}{N} \geq H(\omega) - \omega H\left(\frac{\tau}{\omega}\right) - (1-\omega) H\left(\frac{\tau}{1-\omega}\right) \quad (1)$$

where $\tau = vN^{-1}$.
Hence, there are constant-weight codes which rate satisfies (1). The corresponding family of sets is, according to lemma, t-TSS, Q.E.D.

Previous results:

1. Bound from **[Tracing traitors, 1994]** $R_t(\omega_0) \geq \frac{1}{8t^4}$ with $\omega_0 = \frac{1}{2t^2}$.

2. Bound from **[Gu, Y., Miao, Y. 2016]** $|M(N,\omega)| \geq \frac{\binom{N}{\lceil w/t^2 \rceil}}{\binom{w}{\lceil w/t^2 \rceil}}$

what asymptotically means $R_t(\omega) \geq H(\frac{\omega}{t^2}) - 2\omega H(\frac{1}{t^2})$.

For $t = 2$: theorem proves that $R_2 = \max_\omega R_t(\omega) \geq 0.0181$ and the best known bound is $R_{2,M} = 0.0159$ [Gu, Miao, 2016].

For $t = 3$: the new bound is $R_3 = 0.00316$ and the best known bound is $R_{3,M} = 0.0027$ [Gu, Miao, 2016].

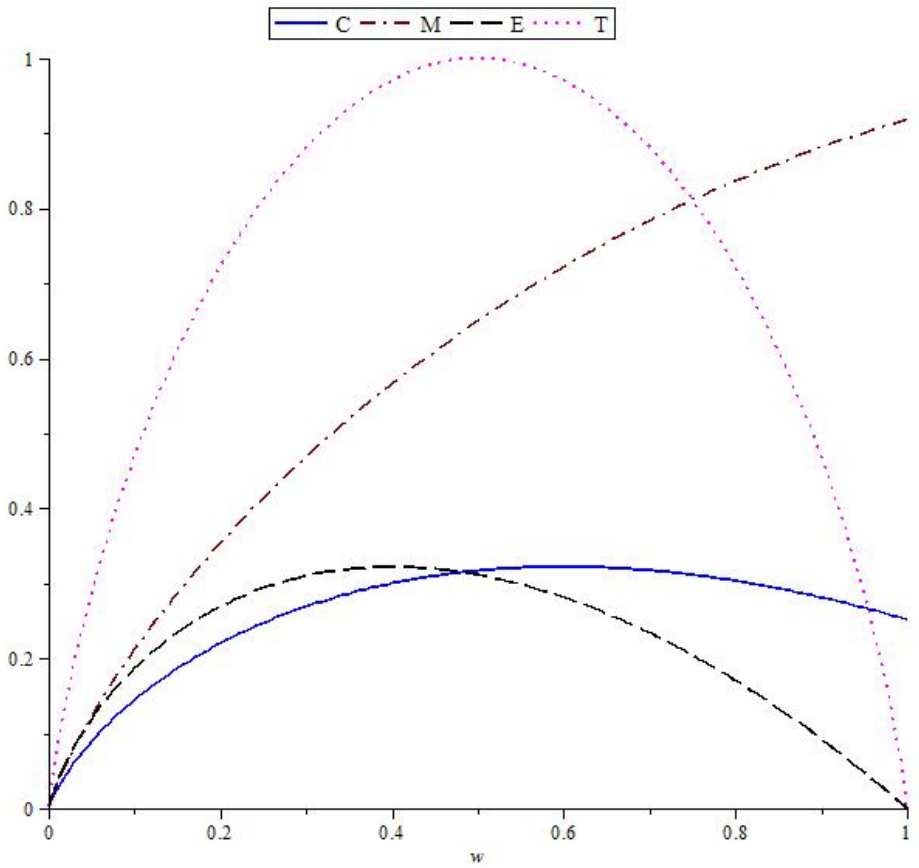
For arbitrary t: since

$$H(\omega) + H(t^{-2}) \geq (1-\omega)H\left(t^{-2}\omega/(1-\omega)\right) + H(t^{-2}\omega),$$

it can be seen that the new bound is always better than previously known ones.

Conclusion

- We proved the new lower bound on the rate of t-TSS codes, which, of course, is the lower bound for t-IPPSS. But we don't know BETTER lower bounds for t-IPPSS.
- It was proved by Gu and Miao in 2016 that t-TSS is t^2-superimposed (cover-free) codes. That gives rather good upper bound for t-TSS. But known upper bounds for IPPSS are rather weak, we think.



Trivial bound: $R_T(\omega) \leq H(\omega)$.

Bound of Collins: $R_C(\omega) \leq H(\omega/3) - 2\omega/3$.

Bound of Miao: $R_M(\omega) \leq H(\omega/3)$

Bound of Erdös: $R_E(\omega) \leq H(\omega/2) - \omega$.

Thank you for your attention!