Bounding the minimum distance of affine variety codes using symbolic computations of footprints

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Tools

Given a field $\mathbb{F}$, an ideal $J \subset \mathbb{F}[X_1, \ldots, X_m]$ and a monomial ordering $\prec$, the footprint is:

$$\Delta_\prec(J) = \{M = X_1^{i_1} \cdots X_m^{i_m} \mid M \text{ is not the leading monomial of any polynomial in } J\}$$

By definition of a Gröbner basis the set $\Delta_\prec(J)$ can be read of from it.

Theorem: $\{M + J \mid M \in \Delta_\prec(J)\}$ is a basis for $\mathbb{F}[X_1, \ldots, X_m]/J$ as a vector space over $\mathbb{F}$.

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The Klein curve

$$I_8 = \langle Y^3 + X^3Y + X, X^8 + X, Y^8 + Y \rangle \subset \mathbb{F}_8[X, Y]$$

Ordering $\prec_w$ is given by $X^\alpha Y^\beta \prec_w X^\gamma Y^\delta$ if either (i) or (ii) holds

$$(i)\ 2\alpha + 3\beta < 2\gamma + 3\delta, \quad (ii)\ 2\alpha + 3\beta = 2\gamma + 3\delta \text{ but } \beta < \delta.$$

$\{Y^3 + X^3Y + X, X^8 - X, X^7Y + Y\}$ is a Gröbner basis for I_8 w.r.t. $\prec_w$.

Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2	X^5Y^2	X^6Y^2	
Y	XY	X^2Y	X^3Y	X^4Y	X^5Y	X^6Y	
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Figure: $\Delta_{\prec_w}(I_8)$

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Affine variety codes

Ideal $I \subset \mathbb{F}_q[X_1, \ldots, X_m]$

$I_q := I + \langle X_1^q - X_1, \ldots, X_m^q - X_m \rangle$

$V(I_q) =: \{P_1, \ldots, P_n\}$

$$\mathrm{ev} : \begin{cases} \mathbb{F}_q[X_1, \ldots, X_m]/I_q \to \mathbb{F}_q^n \\ \mathrm{ev}(F + I_q) = (F(P_1), \ldots, F(P_n)) \end{cases}$$

For $L \subset \Delta_{\prec}(I_q)$ define:

$C(I, L) = \mathrm{Span}_{\mathbb{F}_q}\{\mathrm{ev}(M + I_q) \mid M \in L\}.$

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The footprint bound

Corollary: $\#V(I_q) = \#\Delta_\prec(I_q)$ and $\dim C(I, L) = \#L$.

Proof: We know that $\{M + I_q \mid M \in \Delta_\prec(I_q)\}$ is a basis for $\mathbb{F}_q[X_1, \ldots, X_m]/I_q$ as a vector space. By Lagrange interpolation $\mathrm{ev} : \mathbb{F}_q[X_1, \ldots, X_m]/I_q \to \mathbb{F}_q^n$ is surjective. But I_q is the vanishing ideal of $\{P_1, \ldots, P_n\}$ and therefore ev is injective. □

Corollary: Consider $\vec{c} = \mathrm{ev}(F + I_q)$. Then $w_H(\vec{c}) = n - \#\Delta_\prec(\langle F \rangle + I_q)$.

Proof: Replace I with $\langle F \rangle + I$ in above corollary. □

$\square_\prec = \{M \in \Delta_\prec(I_q) \mid M \in \mathrm{Im}(\langle F \rangle + I_q)\}$ and $w_H(\vec{c}) = \#\square_\prec(F)$.

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First "naive" bound on minimum distance

Let $\vec{c} = \text{ev}(F + I_q)$ and $\text{lm}(F) = XY$.

$$w_H(\vec{c}) = \#\Box_{\prec_w}(F) \geq \#\{XY, X^2Y, \ldots, X^6Y, XY^2, X^2Y^2, \ldots, X^6Y^2\} = 12.$$

Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2	X^5Y^2	X^6Y^2	
Y	XY	X^2Y	X^3Y	X^4Y	X^5Y	X^6Y	
1	X	X^2	X^3	X^4	X^5	X^6	X^7

7	6	5	4	3	2	1	
14	12	10	8	6	4	2	
22	19	16	13	10	7	4	1

Figure: $\Delta_{\prec_w}(I_8)$ and naive bound on $w_H(\vec{c})$ for all possible leading monomial

How to derive improved information on $\#\square(F)$

Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2	X^5Y^2	X^6Y^2	
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6	8	10	12	14	16	18	
3	5	7	9	11	13	15	
0	2	4	6	8	10	12	14

Figure: $\Delta_{\prec_w}(I_8)$ and $w(X^iY^j) = 2i + 3j$.

...if only I and $\prec_w$ satisfied the order domain conditions...but they do not

The order domain conditions

Definition: An ideal I and a weighted degree ordering $\prec_w$ satisfy the order domain condition if:

1. I has a Gröbner basis $\{F_1, \ldots, F_s\}$ with respect to $\prec_w$ such that all F_i possess (exactly) two monomials of highest weight.
2. For $M, N \in \Delta_{\prec_w}(I)$ with $M \neq N$ we have $w(M) \neq w(N)$.

Let $\vec{c} = \text{ev}(F)$ where $\text{lm}(F) = M \in \Delta_{\prec_w}(I_q)$ and $w(M) = \lambda$.

$$\begin{aligned} w_H(\vec{c}) &\geq n - \#\Big(w(\Delta_{\prec_w}(I_q)) \backslash (\lambda + w(\Delta_{\prec_w}(I))\Big) \\ &\geq n - \#\Big(w(\Delta_{\prec_w}(I)) \backslash (\lambda + w(\Delta_{\prec_w}(I)\Big) = n - \lambda \end{aligned}$$

Klein curve $I = \langle Y^3 + X^3 Y + X \rangle$ satisfies (1), but NOT (2).

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Second order domain condition not being satisfied

A family of cases where only first order domain condition is satisfied was treated in [GM15]. However, [GM15] does not apply to $\langle Y^3 + X^3Y + X \rangle$.

The following is involved, but worth it!

Codes from the Klein curve

dim	1	2	3	4	5	6	7	8	9	10	11
d_{naive}	22	19	16	14	13	12	-	10	8	-	7
d_{us}	22	19	18	16	15	-	13	12	-	10	9
d_{grassl}	22	19	18	17	15	14	13	12	11	10	9
dim	12	13	14	15	16	17	18	19	20	21	22
d_{naive}	-	6	5	-	-	4	3	-	2	-	1
d_{us}	-	7	6	5	-	4	3	-	2	-	1
d_{grassl}	8	7	7	6	5	4	4	3	2	2	1

Table: Bounds for the Klein codes: d_{naive} is the naive bound, d_{us} is the involved bound. For comparison d_{grassl} is the best known minimum distance from Grassl's table

Moreover we obtain additional information. For instance the $[22, 21, 1]_8$ code only contains 7 codewords of weight 1.

$F = Y + a_1X + a_2$

Clearly, $\{Y, Y^2, XY, XY^2, \ldots, X^6Y, X^6Y^2\} \subset \Box_{\prec_w}(F)$.

$$Y^2F(X,Y)$$

$$\xrightarrow{Y^3+X^3Y+X} X^3Y + a_1XY^2 + a_2Y^2 + X$$

$$\xrightarrow{F(X,Y)} a_1X^4 + (a_1^3 + a_2)X^3 + a_1^2a_2X^2 + (a_1a_2^2 + 1)X + a_2^3.$$

If $a_1 \neq 0$ then

$$\{X^4, X^5, X^6, X^7\} \subset \Box_{\prec_w}(F).$$

If $a_1 = 0$ and $a_2 \neq 0$ then

$$\{X^3, X^4, X^5, X^6, X^7\} \subset \Box_{\prec_w}(F).$$

If $a_1 = a_2 = 0$ then

$$\{X, X^2, X^3X^4, X^5, X^6, X^7\} \subset \Box_{\prec_w}(F).$$

$w_H(\vec{c}) \geq 14 + \min\{4, 5, 6\} = 18.$

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If $a_1 \neq 1$ then

$$\{X^3Y, X^4Y, X^5Y, X^6Y\} \in \Box_{\prec_w}(F).$$

$$\begin{aligned} & Y((a_1+1)X^3Y + a_2XY^2 + a_3X^2Y + a_4Y^2 + a_5XY + a_6Y + \\ \xrightarrow{F(X,Y)} \ & (a_1+1)(a_1X^6 + a_2X^4Y + a_3X^5 + a_4X^3Y + a_5X^4 + a_6X^3) \\ & +a_2XY^3 + a_3X^2Y^2 + a_4Y^3 + a_5XY^2 + a_6Y^2 + XY. \end{aligned}$$

If $a_1 \neq 0$ then we also have

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$F(X,Y) = Y^2 + a_1X^3 + a_2XY + a_3X^2 + a_4Y + a_5X + a_6$ – cont.

Assuming next that $a_1 = 0$ the above expression becomes

$$\begin{aligned}
& a_2X^4Y + a_3X^5 + a_4X^3Y + a_5X^4 + a_6X^3 + a_2XY^3 \\
& +a_3X^2Y^2 + a_4Y^3 + a_5XY^2 + a_6Y^2 + XY \\
\xrightarrow{Y^3+X^3Y+X} \quad & a_3X^5 + a_4X^3Y + a_5X^4 + a_6X^3 + a_3X^2Y^2 + a_4Y^3 \\
& +a_5XY^2 + a_6Y^2 + XY + a_2X^2 \\
\xrightarrow{F(X,Y)} \quad & a_3X^5 + a_5X^4 + a_6X^3 + a_3a_2X^3Y + a_3^2X^4 + a_3a_4X^2Y \\
& +a_3a_5X^3 + a_3a_6X^2 + a_5XY^2 + a_6Y^2 + XY + a_2X^2.
\end{aligned}$$

If $a_3 \neq 0$ then

$$\{X^5, X^6, X^7\} \subset \Box_{\prec_w}(F).$$

$F(X,Y) = Y^2 + a_1X^3 + a_2XY + a_3X^2 + a_4Y + a_5X + a_6$ – cont.

Hence, continuing under the assumption $a_3 = 0$ we are left with

$$
\begin{aligned}
& a_5X^4 + a_6X^3 + a_5XY^2 + a_6Y^2 + XY + a_2X^2 \\
\xrightarrow{F(X,Y)} \; & a_5X^4 + a_6X^3 + a_5a_2X^2Y + a_5a_4XY + a_5^2X^2 + a_5a_6X + a_6Y^2 \\
& + XY + a_2X^2.
\end{aligned}
$$

and so on ... and so on ...

$w_H(\vec{c}) \geq 7 + \min\{6, 6, 8, 9, 6, 7, 13\} = 13.$

$F(X,Y) = Y^2 + a_1X^3 + a_2XY + a_3X^2 + a_4Y + a_5X + a_6$ – cont.

Hence, continuing under the assumption $a_3 = 0$ we are left with

$$\begin{aligned} & a_5X^4 + a_6X^3 + a_5XY^2 + a_6Y^2 + XY + a_2X^2 \\ \xrightarrow{F(X,Y)} \; & a_5X^4 + a_6X^3 + a_5a_2X^2Y + a_5a_4XY + a_5^2X^2 + a_5a_6X + a_6Y^2 \\ & +XY + a_2X^2. \end{aligned}$$

and so on ... and so on ...

$$w_H(\vec{c}) \geq 7 + \min\{6, 6, 8, 9, 6, 7, 13\} = 13.$$

Final comparison

Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2	X^5Y^2	X^6Y^2	
Y	XY	X^2Y	X^3Y	X^4Y	X^5Y	X^6Y	
1	X	X^2	X^3	X^4	X^5	X^6	X^7

7	6	5	4	3	2	1	
14	12	10	8	6	4	2	
22	19	16	13	10	7	4	1

13	10	7	5	3	2	1	
18	15	12	9	6	4	2	
22	19	16	13	10	7	4	1

Figure: $\Delta_{\prec_w}(I_8)$, naive bound on $w_H(\vec{c})$ and improved information

Thanks!